Introduction to Probability, Statistics and Random Processes

Chapter 8: Statistical inference with classical methods

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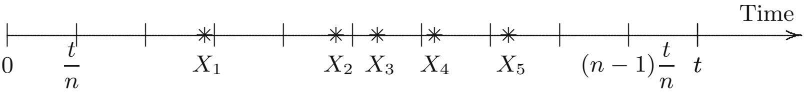
Poisson distribution, review

Example

Calls arriving at a telephone exchange — the exchange is connected to a large number of people who make phone calls now and then.

Telephone calls arrive at random times $X_1, X_2, \cdots$ *at the telephone exchange during a time interval* $[0, t]$.



- The rate λ at which arrivals occur is constant over time
- Consider the total number of calls in an interval $[0, t]$, denoted by $N([0, t])$, abbreviating to N_t
- $N_t = R_1 + R_2 + \cdots + R_n$ has a $Bin(n, p)$ distribution, with $p = \lambda t / n$
- $P(N_t = k) = \binom{n}{k}(\frac{\lambda t}{n})^k(1 - \frac{\lambda t}{n})^{n-k}$ for $k = 0, \cdots, n$
- $\lim\limits_{n\to\infty} P(N_t = k) = \lim\limits_{n\to\infty} \binom{n}{k}\frac{1}{n^k}\cdot(\lambda t)^k\cdot(1-\frac{\lambda t}{n})^n\cdot(1-\frac{\lambda t}{n})^{-k} = \frac{(\lambda t)^k}{k!}e^{-\lambda t}$

Poisson distribution, review

Definition

A discrete random variable X has a Poisson distribution with parameter μ, where $\mu > 0$ if its probability mass function p is given by

$$p(k) = P(X = k) = \frac{\mu^k}{k!} e^{-\mu} \quad \text{for} \quad k = 0, 1, 2, \cdots$$

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Remark

Let λ be the intensity of occurrence

$$\mu = \lambda \cdot t$$

Poisson distribution, review

quick exercise

Consider the event "exactly one call arrives in the interval *[0,2s]*." The probability of this event is

$$P(N_{2s} = 1) = \lambda \cdot 2s \cdot e^{-\lambda 2s}$$

But note that this event is the same as "there is exactly one call in the interval *[0,s)* and no calls in the interval *[s,2s]*, or no calls in *[0,s)* and exactly one call in *[s,2s]*." Verify that you get the same answer if you compute the probability of the event in this way.

Poisson distribution, review

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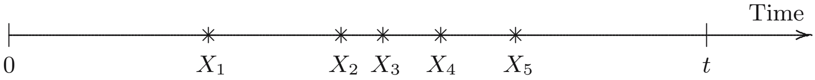
hint: the probability of exactly one call in *[0,s)* and no calls in *[s,2s]* equals

$$\begin{aligned} P(N([0,s)) = 1, N([s,2s]) = 0) &= P(N([0,s)) = 1)P(N([s,2s]) = 0) \\ &= P(N([0,s)) = 1)P(N([0,s]) = 0) \\ &= \lambda s e^{-\lambda s} \cdot e^{-\lambda s} \end{aligned}$$

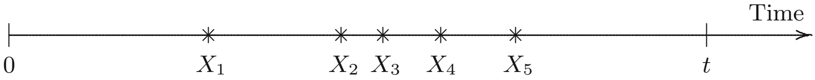
Interarrival times



- The differences $T_i = X_i - X_{i-1}$ are called interarrival times
- Define $T_1 = X_1$, the time of the first arrival
- Event $T_1 > t$: the first call arrives after t, that is $N_t = 0$ (no calls in $[0, t]$)

$$P(T_1 \leq t) = 1 - P(T_1 > t) = 1 - P(N_t = 0) = 1 - e^{-\lambda t}.$$

Interarrival times



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- T_1 has an exponential distribution with parameter λ.

Interarrival times

- consider the conditional probability that $T_2 > t$, given that $T_1 = s$

$$
\begin{aligned}
P(T2 > t | T1 = s) &= P(\text{no arrivals in } (s, s+t] | T_1 = s) \\
&= P(\text{no arrivals in } (s, s+t]) \\
&= P(N((s, s+t]) = 0) = e^{-\lambda t}.
\end{aligned}
$$

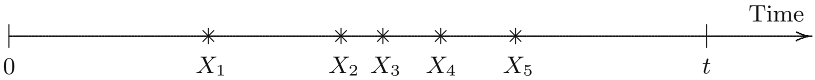
- Hence $P(T_2 > t) = e^{-\lambda t}$, $T_2 \sim Exp(\lambda)$
- Analogously, $T_i \sim Exp(\lambda)$

Interarrival times

Definition

The one-dimensional Poisson process with intensity λ is a sequence X_1, X_2, X_3, λ of random variables having the property that the interarrival times $X1$, $X_2 - X_1$, $X_3 - X_2, \cdots$ are independent random variables, each with an $Exp(\lambda)$ distribution.

Interarrival times



- What is the distribution of X_i?
- X_i is a sum of i independent exponentially distributed random variables

The points of Poisson process

For $i = 1, 2, \cdots$, the random variable X_i has a $Gam(i, \lambda)$ distribution.

The distribution of points

- if we know that n points are generated in an interval, where do these points lie?
- Let this interval be $[0, a]$.
- We start with the simplest case, $N([0, a]) = 1$

$$
\begin{aligned}
P(X_1 \le s | N([0,a]) = 1) &= \frac{P(X_1 \le s, N([0,a]) = 1)}{P(N([0,a]) = 1)} \\
&= \frac{P(N([0,s]) = 1, N((s,a]) = 0)}{P(N([0,a]) = 1)} \\
&= \frac{\lambda s \cdot e^{-\lambda s} \cdot e^{-\lambda(a-s)}}{\lambda a \cdot e^{-\lambda a}} \\
&= \frac{s}{a}.
\end{aligned}
$$

- X_1 is uniformly distributed over the interval $[0, a]$.
- $\cdots$

The distribution of points

Location of the points, given their number.

Given that the Poisson process has n points in the interval $[a, b]$, the locations of these points are independently distributed, each with a uniform distribution on $[a, b]$.