Introduction to Probability, Statistics and Random Processes

Chapter 1: Basic Concepts

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Set Theory

- A **set** is a collection of things called **elements**.
- A set is denoted in capital letters and defined by simply listing its elements in curly brackets. Example: $A = \{b, c\}$.
- Can also be defined as $A = \{\text{x:x satisfies some property}\}$.
- Ordering does not matter in sets. Thus $\{1, 2, 3, 4\}$ and $\{3, 2, 1, 4\}$ are the same set.
- $b \in A$ - b belongs to A where $\in$ means belongs to.
- And $d \notin A$, where $\notin$ means does not belong.

Important Sets

- The set of natural numbers, $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$
- The set of integers, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- The set of rational numbers $\mathbb{Q}$.
- The set of real numbers $\mathbb{R}$ and the set of complex numbers $\mathbb{C}$.
- Closed intervals on the real line. Example: [2,3] is set of real numbers such that $2 \leq x \leq 3$.
- Open intervals on the real line. Example: (1,2) is the set of real numbers such that $1 < x < 2$.

More on Sets

- Set A is a **subset** of set B if every member of A is also a member of B. We write $A \subset B$, where $\subset$ indicates subset.
- Equivalently B is the **superset** of A, $B \supset A$.
- Two sets are **equal** $A = B$, if they contain the same elements, that is $A \subset B$ and $B \subset A$
- The **universal set** S or Ω is the set of all things that we could possibly consider in the context we are studying.
- The universal set in probability is also called the **sample space**.
- The set with no elements is called the **empty** or **null set** $\phi = \emptyset$.

Venn Diagrams

- Venn Diagrams are very useful in visualizing relations between sets.
- In Venn Diagrams, a set is depicted by a closed region.

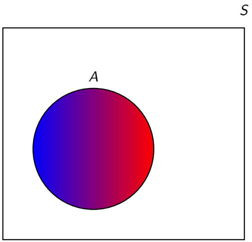

Figure: Venn Diagram

Venn Diagrams

- The figure below shows two sets, A and B, where $B \subset A$.
- Both A and B are subsets of the universal set S.

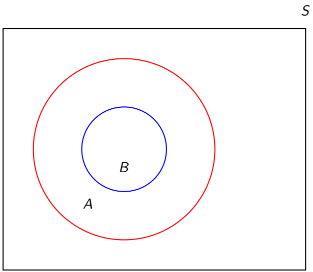

Figure: Venn Diagram for two sets A and B, where $B \subset A$.

Set Operations: Union

- The union of two sets is a set containing all elements that are in A or in B.
- Example: $\{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$.
- In general the union of n sets $A_1, A_2, \dots, A_n$ is represented as $\bigcup_{i=1}^{n} A_i$.

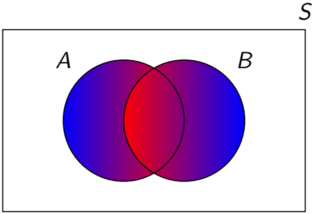

Figure: The shaded area shows the set $B \cup A$.

Set Operations: Intersection

- The intersection of two sets A and B is a set containing all elements that are in A and B.
- Example: $\{1,2\} \cap \{2,3\} = \{2\}$.
- In general, the intersection of n sets $\bigcap_{i=1}^{n} A_i$ is the set consisting of elements that are in all n sets.

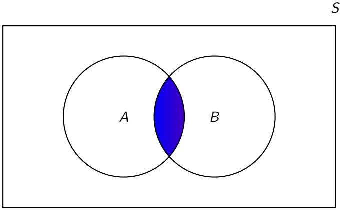

Figure: The shaded area shows the set $B \cap A$.

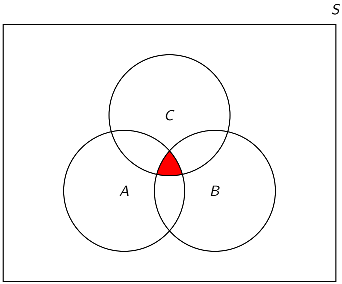

Figure: The shaded area shows the set $A \cap B \cap C$.

Set Operations: Complement

- The complement of a set A is the set of all elements that are in the universal set S but not in A.

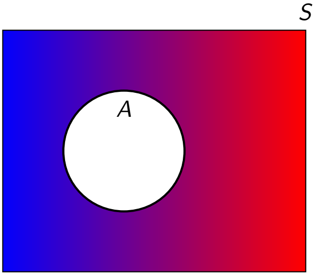

Figure: The shaded area shows the set $\bar{A} = A^c$.

Set Operations: Difference (Subtraction)

- The set $A - B$ consists of elements that are in A but not in B.
- Example: $A = \{1, 2, 3\}$ and $B = \{3, 5\}$, then $A - B = \{1, 2\}$

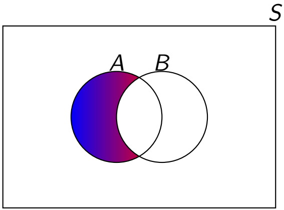

Figure: The shaded area shows the set $A - B$.

Mutually Exclusive or Disjoint Sets

- Sets A and B are mutually exclusive or disjoint if they do not have any shared elements.
- The intersection of two sets that are disjoint is the empty set i.e. $A \cap B = \emptyset$.
- In general, several sets are disjoint if they are pairwise disjoint.

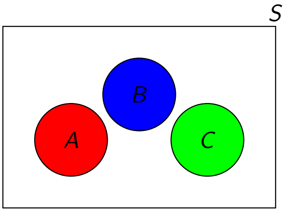

Figure: Sets A, B, and C are disjoint.

Partition of Sets

- A collection of non-empty set $A_1, A_2, \ldots$ is a **partition** of A if they are disjoint and their union is A.

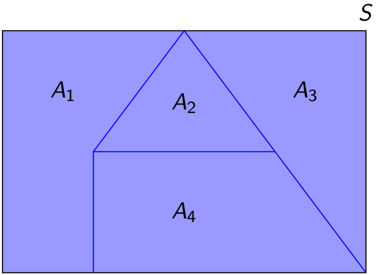

Figure: The collection of sets A_1, A_2, A_3 and A_4 is a partition of S.

Important Theorems

- De Morgan's Law:

  For any sets $A_1, A_2, \ldots, A_n$, we have:
  - $(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n)^c = {A_1}^c \cap {A_2}^c \cap {A_3}^c \cap \ldots \cap {A_n}^c$
  - $(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n)^c = {A_1}^c \cup {A_2}^c \cup {A_3}^c \cup \ldots \cup {A_n}^c$

- Distributive Law

  For any sets
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Cardinality in Finite sets

- **Cardinality** is basically the size of the set.
- If set A only has a finite number of elements, its cardinality is simply the number of elements in A.
- For example, if $A = \{2, 4, 6, 8, 10\}$, then $\mid A \mid = 5$.
- We will discuss cardinality of infinite sets later.

Inclusion-Exclusion Principle

- The inclusion-exclusion principle states that for two finite sets A, B and C.
  - $|A \cup B| = |A| + |B| - |A \cap B|$,
  - $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.
- In general for n finite sets $A_1, A_2, \ldots, A_n$

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n| .$$

Cardinality in Infinite Sets

- There are two kinds of infinite sets: countable sets and uncountable sets.
- The difference between the two is that you can list elements in a countable set, so $A = \{a_1, a_2, \ldots\}$, but you cannot list elements in an uncountable set.
- The set $\mathbb{R}$ is uncountable and much *larger* than countably infinite sets $\mathbb{N}$ and $\mathbb{Z}$.

Countable vs. Uncountable Sets

- A more rigorous definition of a countable set A is
  - if it is a finite set, $|A| < \infty$; or
  - it can be put in one-to-one correspondence with natural numbers $\mathbb{N}$, in which case the set is said to be countably infinite.
- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and any of their subsets are countable.
- Any set containing an interval on the real line such as $[a, b], (a, b], [a, b)$ and (a, b), where $a < b$ is uncountable.

Useful Theorems on Countability

- Any subset of a countable set is countable. Any superset of an uncountable set is uncountable.
- If $A_1, A_2, \cdots$ is a list of countable sets, then the set $\bigcup_i A_i = A_1 \cup A_2 \cup A_3 \cdots$ is also countable.
- If A and B are countable, then $A \times B$ is also countable.

The Cartesian Product

- The Cartesian Product of two sets A and B, written as $A \times B$, is the set containing ordered pairs from A and B.
- Thus $A \times B = \{(x, y) \mid x \in \text{A and y} \in B\}$
- For example, if $A = \{1, 2, 3\}$ and $B = \{H, T\}$, then $A \times B = \{(1, H), (1, T), (2, H), (2, T), (3, H), (3, T)\}$
- It is important to note that the pairs are ordered, thus $(1, H) \neq (H, 1)$ and $A \times B \neq B \times A$.

Multiplication Principle

- **Multiplication principle**: If two finite sets A has M elements and B has N elements, then $A \times B$ has $M \times N$ elements.
- In general for sets $A_1, A_2, \ldots A_n$ with $|A_1| = M_1$,$|A_2| = M_2$,...,$|A_n| = M_n$, we have $|A_1 \times A_2 \times \ldots \times A_n| = M_1 \times M_2 \times \ldots \times M_n$.
- An important example is $\mathbb{R}^n$ where n is a natural number. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is set of all points in the 2-D plane.

Functions

- A **function** maps elements from the **domain** set to elements in another set called the **co-domain**.
- Each input in the domain is mapped to exactly one output in the co-domain.
- It is denoted as $f : A \to B$.
- The **range** of a function is the set of all possible values of $f(x)$ and is a subset of the co-domain.

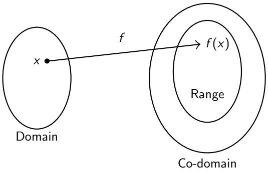

Figure: Function $f : A \to B$, the range is always a subset of the co-domain.