Introduction to Probability, Statistics and Random Processes

Chapter 4: Continuous and Mixed Random Variables

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Uniform Distribution

Example

Suppose we measure for a long time the emission of radioactive particles of some material. Suppose that the experiment consists of recording in each hour at what times the particles are emitted. Then the outcomes will lie in the interval *[0,60]* minutes. If the measurements would concentrate in any way, there is either something wrong with your Geiger counter or you are about to discover some new physical law. Not concentrating in any way means that subintervals of the same length should have the same probability. It is then clear that the probability density function associated with this experiment should be constant on *[0, 60]*.

Uniform Distribution

A continuous random variable X is said to have a *Uniform* distribution over the interval $[a, b]$, shown as $X \sim Uniform(a, b)$, if its PDF is given by

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & x < a \text{ or } x > b \end{cases}$$

Uniform Distribution

- CDF of $X \sim Uniform(a, b)$ is given by

$$F(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } a \leq x \leq b \\ 1 & \text{for } x > b \end{cases}$$

- Expectation of X is given by

$$E[X] = \frac{a+b}{2}$$

- Variance of X is given as

$$Var(X) = E[X^2] - (E[X])^2 = \frac{(b-a)^2}{12}$$

Uniform Distribution

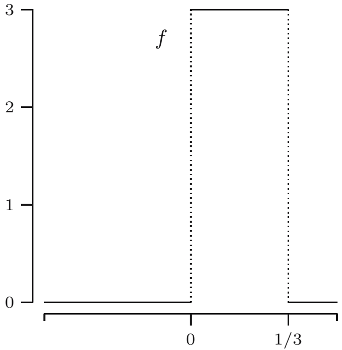

Figure: The probability density function and the distribution function of the *Uniform(0,1/3)* distribution.

Uniform Distribution

- PDF of $X \sim Uniform(a, b)$.

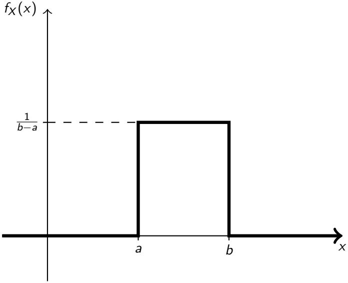

Figure: PDF for a continuous random variable uniformly distributed over (a, b).

Exponential Distribution

Example

Recall the chemical reactor example: Let v be the effluent volumetric flow rate, i.e., the volume that leaves the reactor over a time interval $[0,t]$ is vt (and an equal volume enters the vessel at the other end). Let $\mathtt{V}$ be the volume of the reactor vessel. Then in total a fraction $(v/V)t$ will have left the vessel during $[0,t]$, when t is not too large. Let the random variable T be the residence time of a particle in the vessel.

- Find the distribution of T.

Exponential Distribution

- Find the distribution of T.
  - divide the interval *[0,t]* in n small intervals of equal length t/n
  - (perfect mixing, the particle's position is uniformly distributed over the volume) the particle has probability $p = (v/V)t/n$ to have left the vessel during any of the n intervals of length t/n
  - the behavior of the particle in different time intervals of length t/n is independent
  - if we call "leaving the vessel" a success, then T has a geometric distribution with success probability p

$$P(T > t) = \lim_{n\to\infty}(1 - \frac{vt}{V} \cdot \frac{1}{n})^n = e^{-\frac{v}{V}t}.$$

Exponential Distribution

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$$P(T > t) = \lim_{n\to\infty}(1 - \frac{vt}{V} \cdot \frac{1}{n})^n = e^{-\frac{v}{V}t}.$$

  - Thus $F(t) = 1 - e^{-\frac{v}{V}t}$, $f(t) = \frac{d}{dt}(1 - e^{-\frac{v}{V}t}) = \frac{v}{V}e^{-\frac{v}{V}t}$ for $t \geq 0$.

Exponential Distribution

- Widely used in different applications to model the time elapsed between events.

A continuous random variable X is said to have an *exponential* distribution with parameter $\lambda > 0$, shown as $X \sim \mathit{Exponential}(\lambda)$, if its PDF is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Exponential Distribution

- CDF of $X \sim Exponential(\lambda)$ is

$$F(x) = (1 - e^{-\lambda x}) \text{ for } x \geq 0.$$

- Expectation of X is given by

$$EX = \frac{1}{\lambda}$$

- Variance of X is

$$Var(X) = EX^2 - (EX)^2 = \frac{1}{\lambda^2}$$

Exponential Distribution

- PDF of $X \sim Exponential(\lambda)$ for $\lambda = 1, 2, 3$.

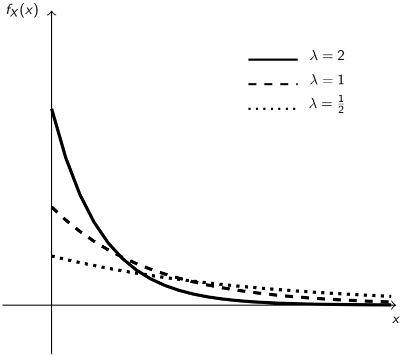

Figure: PDF of the exponential random variable

Exponential Distribution: PDF and CDF

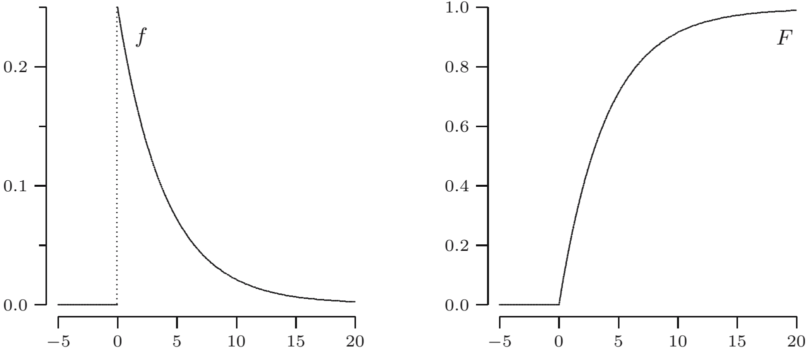

Figure: The probability density and the distribution function of the *Exp(0.25)* distribution.

Exponential Distribution

If X is exponential with parameter $\lambda > 0$, then X is a *memoryless* random variable, that is

$$P(X > s + t | X > s) = P(X > t), \text{ for } s, t \geq 0.$$

- follows directly from

$$P(X > s+t | X > s) = \frac{P(X > s + t)}{P(X > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X > t).$$

Exponential Distribution

If X is exponential with parameter $\lambda > 0$, then X is a *memoryless* random variable, that is

$$P(X > s + t | X > s) = P(X > t), \text{ for } s, t \geq 0.$$

- Say X is the waiting time until arrival of customer.
- From the memoryless property we have that, it does not matter how long you have waited.
- If you have not observed a customer until time s, the distribution of waiting time (from time s) is the same as when starting from at time 0.

Normal(Gaussian) Distribution

- By far the most important probability distribution.
- We will see it's importance in the Central Limit Theorem later on.
  - *The normal distribution is an important tool to approximate the probability distribution of the average of independent random variables.*

Normal Distribution

A continuous random variable Z is said to be a *standard normal (standard Gaussian)* random variable, shown as $Z \sim N(0,1)$, if its PDF is given by

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad \text{for all } z \in \mathbb{R}.$$

- If $Z \sim N(0,1)$, then expectation $EZ = 0$ and variance $Var(Z) = 1$.

Normal Distribution

- PDF of the standard normal random variable.

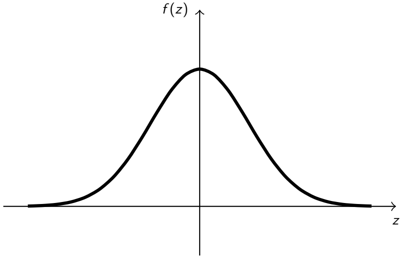

Figure: PDF of the standard normal random variable

Normal Distribution

The CDF of the standard normal distribution is denoted by the Φ function:

$$\Phi(x) = P(Z \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du.$$

- The integral in the CDF does not have a closed form solution.
- Due to it's importance, the values of $F(z)$ have been calculated and readily available.

Normal Distribution

- CDF of a standard normal variable.

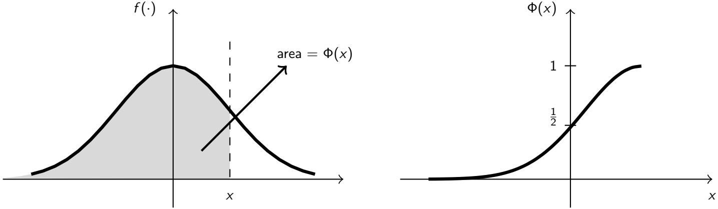

Figure: The Φ function (CDF of standard normal).

Normal Distribution

- Properties of the ϕ function include
  1. $\lim_{x\to\infty} \Phi(x) = 1, \quad \lim_{x\to-\infty} \Phi(x) = 0;$
  2. $\Phi(0) = \frac{1}{2};$
  3. $\Phi(-x) = 1 - \Phi(x)$, for all $x \in \mathbb{R}$.
- A very useful bound that we can use is

$$\frac{1}{\sqrt{2\pi}} \frac{x}{x^2+1} e^{-\frac{x^2}{2}} \leq 1 - \Phi(x) \leq \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-\frac{x^2}{2}}$$

Normal Distribution

- From a standard normal random variable, we can obtain any normal variable by shifting and scaling.

If Z is a standard normal random variable and $X = \sigma Z + \mu$, then X is a normal random variable with mean μ and variance σ^2, i.e,

$$X \sim N(\mu, \sigma^2).$$

Normal Distribution

If X is a normal random variable with mean μ and variance σ^2, i.e, $X \sim N(\mu, \sigma^2)$, then

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

$$F(x) = P(X \leq x) = \Phi\left(\frac{x-\mu}{\sigma}\right),$$

$$P(a < X \leq b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$$

Normal Distribution

- PDFs for normal distributions with different mean and variance.

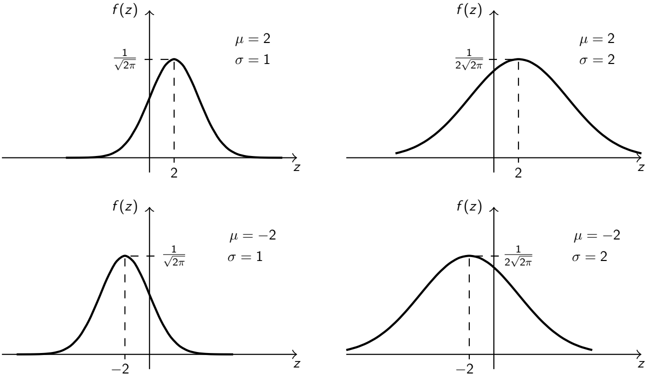

Figure: PDF for normal distribution

Normal Distribution

- An important property is that a linear transformation of a normal random variable is itself a normal random variable.
- We thus have the following theorem.

Theorem: If $X \sim N(\mu_X, \sigma_X^2)$, and $Y = aX + b$, where $a, b \in \mathbb{R}$, then $Y \sim N(\mu_Y, \sigma_Y^2)$ where

$$\mu_Y = a\mu_X + b, \;\; \sigma_Y^2 = a^2\sigma_X^2.$$