GRAPHICAL MODELS AND IMAGE PROCESSING
Vol. 59, No. 3, May, pp. 164–172, 1997
ARTICLE NO. IP970422

3D Well-Composed Pictures

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Received July 25, 1995; revised January 31, 1997; accepted February 19, 1997

By a *segmented* image, we mean a digital image in which each point is assigned a unique label that indicates the object to which it belongs. By the foreground (objects) of a segmented image, we mean the objects whose properties we want to analyze and by the background, all the other objects of a digital image. If one adjacency relation is used for the foreground of a 3D segmented image (e.g., 6-adjacency) and a different relation for the background (e.g., 26-adjacency), then interchanging the foreground and the background can change the connected components of the digital picture. Hence, the choice of foreground and background is critical for the results of the subsequent analysis (like object grouping), especially in cases where it is not clear at the beginning of the analysis what constitutes the foreground and what the background, since this choice immediately determines the connected components of the digital picture. A special class of segmented digital 3D pictures called "well-composed pictures" will be defined. Well-composed pictures have very nice topical and geometrical properties; in particular, the boundary of every connected component is a Jordan surface and there is only one type of connected component in a well-composed picture, since 6-, 14-, 18-, and 26-connected components are equal. This implies that for a well-composed digital picture, the choice of the foreground and the background is not critical for the results of the subsequent analysis. Moreover, a very natural definition of a continuous analog for well-composed digital pictures leads to regular properties of surfaces. This allows us to give a simple proof of a digital version of the 3D Jordan–Brouwer separation theorem. © 1997 Academic Press

1. INTRODUCTION

In this paper 3D well-composed pictures are defined and their properties are analyzed. Their definition is based on the concept of a continuous analog. There are actually two different approaches to define a continuous analog of a digital picture. In Artzy *et al.* [2], Herman [9], and Rosenfeld *et al.* [18], a point of a 3D digital image is interpreted as a unit cube in $\mathbb{R}^3$, digital objects are interpreted as connected sets of cubes, and the surface of an object is the set of faces of the cubes that separate the object from its complement. For example, in Fig. 1a a continuous analog of the eight-point digital set is the union of the eight cubes. In the graph interpretation of a digital image, a face in a surface of a continuous analog corresponds to a pair of 6-adjacent points (p, q), where p belongs to the object and q belongs to its complement [9]. A different approach is taken in Kong and Roscoe [10], where, for example, a cube belongs to a continuous analog of a (6, 26) binary digital picture if all of its eight corners belong to the digital object (set of black points), and a face of a cube belongs to the surface of a continuous analog of a digital object if the four corners of the face belong to the boundary of the digital object. For example, in Fig. 1b such a continuous analog of the eight-point digital set is the single cube that has the eight points as its corners. If we treat the corners of faces as points of a (6, 26) digital picture, then the corresponding digital surface is composed of picture points. Such surfaces are analyzed in Morgenthaler and Rosenfeld [14], Kong and Roscoe [10], Francon [6], and Chen and Zhang [3, 4].

We will interpret $\mathbb{Z}^3$ as the set of points with integer coordinates in 3D space $\mathbb{R}^3$. We will denote the set of the closed unit upright cubes which are centered at points of $\mathbb{Z}^3$ by $\mathscr{C}$, and the set of closed faces of cubes in $\mathscr{C}$ by $\mathscr{F}$, i.e., each $f \in \mathscr{F}$ is a unit closed square in $\mathbb{R}^3$ parallel to one of the coordinate planes.

A three-dimensional digital set (i.e., a finite subset of $\mathbb{Z}^3$) can be identified with a union of upright unit cubes which are centered at its points. This gives us an intuitive and simple correspondence between points in $\mathbb{Z}^3$ and cubes in $\mathbb{R}^3$. Since this correspondence plays an important role in this paper, we will describe it formally.

The *continuous analog* $\mathrm{CA}(p)$ of a point $p \in \mathbb{Z}^3$ is the closed unit cube centered at this point with faces parallel to the coordinate planes. The continuous analog of a digital set X (i.e., $X \subseteq \mathbb{Z}^3$) is defined as $\mathrm{CA}(X) = \bigcup\{\mathrm{CA}(x) : x \in X\}$ (see Fig. 1a). Formally, CA is a function $\mathrm{CA} : \mathscr{P}(\mathbb{Z}^3) \to \mathscr{P}(\mathbb{R}^3)$. In particular, we have $\mathscr{C} = \{\mathrm{CA}(p) : p \in \mathbb{Z}^3\}$.

We also define a dual function $\mathrm{Dig}_\in$ to CA which we call *subset* (or *element*) *digitization*: $\mathrm{Dig}_\in : \mathscr{P}(\mathbb{R}^3) \to \mathscr{P}(\mathbb{Z}^3)$ is given by $\mathrm{Dig}_\in(Y) = \{p \in \mathbb{Z}^3 : p \in Y\}$. Clearly, we have $\mathrm{Dig}_\in(\mathrm{CA}(X)) = X$ for every $X \subseteq \mathbb{Z}^3$. The equation

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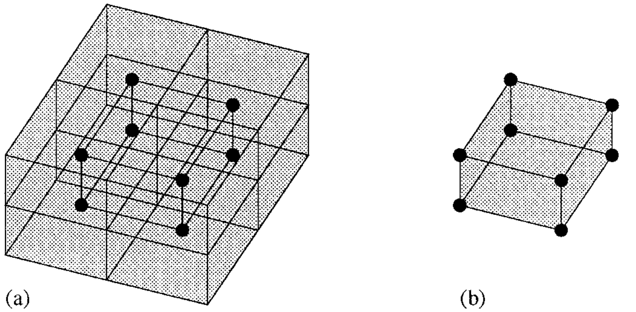

FIG. 1. There are actually two different approaches to define a continuous analog of a digital picture.

$\mathrm{CA}(\mathrm{Dig}_{\in}(Y)) = Y$ holds only if $Y \subseteq \mathbb{R}^3$ is a union of some cubes in $\mathscr{C}$.

We will define a 3D digital picture as well-composed if the boundary surface of its continuous analog is a 2D manifold (i.e., it "looks" locally like a planar open set). This definition implies a simple correspondence between a 3D digital image and the boundary surface of its continuous analog when digital objects are identified with unions of cubes centered at their points. Thus, we can use well-known properties of continuous boundary surfaces, like the Jordan–Brouwer separation theorem, to determine and analyze properties of these digital images. Additionally, since we will study boundary surfaces, some of our results also apply to surfaces spanned on boundary points of digital pictures. For example, conditions given in Theorem 4.2 also apply to the simple closed surfaces in the Morgenthaler and Rosenfeld sense.

Since we identify cubes (voxels) with points in $\mathbb{Z}^3$ at which they are centered, the following definitions apply as well for cubes in $\mathbb{R}^3$ as for points in $\mathbb{Z}^3$.

Two distinct points $p, q \in \mathbb{Z}^3$ are said to be *face-adjacent* if cubes $\mathrm{CA}(p)$ and $\mathrm{CA}(q)$ share a face, or equivalently, if two of the coordinates of p, q are the same and the third coordinates differ by 1. Two distinct points $p, q \in \mathbb{Z}^3$ are said to be *edge-adjacent* if cubes $\mathrm{CA}(p)$ and $\mathrm{CA}(q)$ share an edge but not a face (i.e., $\mathrm{CA}(p) \cap \mathrm{CA}(q)$ is a line segment), or equivalently, if one of the coordinates of p, q is the same and the other two coordinates differ by 1. Two distinct points $p, q \in \mathbb{Z}^3$ are said to be *corner-adjacent* if cubes $\mathrm{CA}(p)$ and $\mathrm{CA}(q)$ share a vertex but not an edge (i.e., $\mathrm{CA}(p) \cap \mathrm{CA}(q)$ is a single point), or equivalently, if all three of the coordinates of p, q differ by 1.

Two points are said to be *6-adjacent* (*6-neighbors*) if they are face-adjacent, *18-adjacent* (*18-neighbors*) if they are face- or edge-adjacent, and *26-adjacent* (*26-neighbors*) if they are face-, edge-, or corner-adjacent. A set $X \subset \mathbb{Z}^3$ is k-adjacent to a point $p \in \mathbb{Z}^3$ if there exists $q \in X$ such that p and q are k-adjacent, where $k = 6, 18, 26$.

$\mathscr{N}_k(p)$ denotes the set containing $p \in \mathbb{Z}^3$ and all points k-adjacent to p and $\mathscr{N}_k^*(p)$ denotes $\mathscr{N}_k(p)\backslash\{p\}$, where $k = 6, 18, 26$. $\mathscr{N}_{26}(p)$ is also referred to as $\mathscr{N}(p)$ and is called the *neighborhood* of p, whereas $\mathscr{N}_{26}(p)\backslash\{p\}$ is referred to as $\mathscr{N}^*(p)$.

A common *face* of two cubes centered at points $p, q \in \mathbb{Z}^3$ (i.e., a unit square parallel to one of the coordinate planes) can be identified with the pair (p, q). Such pairs are called "surface elements" in Herman [9], since they are constituent parts of object surfaces. We can extend CA to apply also to pairs of points by defining $\mathrm{CA}((p, q)) = \mathrm{CA}(p) \cap \mathrm{CA}(q)$ for $p, q \in \mathbb{Z}^3$ and $\mathrm{CA}(B) = \bigcup\{\mathrm{CA}(x) : x \in B\}$, where B is a set of pairs of points in $\mathbb{Z}^3$. In particular, we have $\mathscr{F} = \{\mathrm{CA}((p, q)) : p, q \in \mathbb{Z}^3$ and p is 6 adjacent to $q\}$.

The (face) *boundary* of a continuous analog $\mathrm{CA}(X)$ of a digital set $X \subseteq \mathbb{Z}^3$ is defined as the union of the set of closed faces each of which is the common face of a cube in $\mathrm{CA}(X)$ and a cube not in $\mathrm{CA}(X)$. Observe that the face boundary of $\mathrm{CA}(X)$ is just the topological boundary $\mathrm{bdCA}(X)$ in $\mathbb{R}^3$. The face boundary $\mathrm{bdCA}(X)$ can also be defined using only cubes of the set $\mathrm{CA}(X)$ as the union of the set of closed faces each of which is a face of exactly one cube in $\mathrm{CA}(X)$. We have $\mathrm{bdCA}(X) = \mathrm{bdCA}(X^c)$, where $X^c = \mathbb{Z}^3\backslash X$ is the complement of X. The (6-) boundary of a digital set $X \subseteq \mathbb{Z}^3$ can be defined as the set of pairs

$$\mathrm{bd}_6 X = \{(p, q) : p \in X \text{ and } q \notin X \text{ and } p \text{ is 6 adjacent to } q\}.$$

We have $\mathrm{bdCA}(X) = \mathrm{CA}(\mathrm{bd}_6 X) = \mathrm{CA}(\mathrm{bd}_6(X^c))$.

Two distinct faces $f_1, f_2 \in \mathscr{F}$ are edge-adjacent if they share an edge, i.e., if $f_1 \cap f_2$ is a line segment in $\mathbb{R}^3$. Two distinct faces f_1, f_2 are corner-adjacent if they share a vertex but not an edge, i.e., if $f_1 \cap f_2$ is a single point in $\mathbb{R}^3$.

In Latecki *et al.* [12] a special class of subsets of 2D binary digital pictures called "well-composed pictures" is defined. The idea is not to allow the "critical configuration" shown in Fig. 2 to occur in a digital picture. Note that this critical configuration can be detected locally.

The 2D well-composed pictures have very nice topological properties; for example, the Jordan curve theorem holds for them, their Euler characteristic is locally computable, and they have only one connectedness relation, since 4-connectedness and 8-connectedness are equivalent. Therefore, when we restrict our attention to well-composed pictures, a number of very difficult problems in digital geometry, as well as complicated algorithms, become

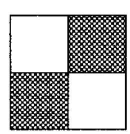

FIG. 2. Critical configuration for non-well-composed 2D pictures.

relatively simple. This is demonstrated in Latecki *et al.* [12] on the example of thinning algorithms. There are practical advantages in applying thinning algorithms to well-composed pictures. The thinning process (sequential as well as parallel) is greatly simplified. We proved that the skeletons obtained are "one-point thick." Thus, the problems with irreducible "thick" skeletons disappear. On the other hand, if a set lacks the property of being well-composed, the digitization process that gave rise to it must not have been topology preserving, since the results in Gross and Latecki [8] show that if the resolution of a digitization process is fine enough to ensure the topology preservation, then the segmented 2D image must be well-composed.

An important motivation for 2D well-composed pictures were connectivity paradoxes which occur if only one adjacency relation (e.g., 4-adjacency) is used in the whole picture. Such paradoxes are pointed out in Rosenfeld and Pfaltz [15] (see also Kong and Rosenfeld [11]). The most popular solution was the idea of using different adjacency relations for the foreground and the background: 8-adjacency for black points and 4-adjacency for white points, or vice versa (first recommended in Duda *et al.* [5]). Rosenfeld [16] developed the foundations of digital topology based on this idea and showed that the Jordan curve theorem is then satisfied. However, the solution with two different adjacency relations does not work if one wants to distinguish more than two colors, i.e., to distinguish among different objects in a segmented image, as shown in Latecki [13]. The same paradoxes appear in 3D segmented images. In the following we will define and analyze 3D segmented "well-composed pictures" in which the connectivity paradoxes do not occur.

2. DEFINITION OF 3D WELL-COMPOSED PICTURES

We will interpret $\mathbb{Z}^3$ as the set of points with integer coordinates in 3D space $\mathbb{R}^3$. We denote by $(\mathbb{Z}^3, X)$, where $x \subseteq \mathbb{Z}^3$, a *binary digital picture* $(\mathbb{Z}^3, \lambda)$, where $\lambda : \mathbb{Z}^3 \to \{0, 1\}$ is given by $\lambda(p) = 1$ iff $p \in X$. We assume that either X or its complement X^c is finite and nonempty.

A binary digital picture is obtained from a segmented picture if some set of points X is distinguished (e.g., points of the same color), which is treated as the *foreground*, and all the other points are lumped together to form the *background*. Usually, each point in X is assigned value 1 (i.e., black) and each point in X^c is assigned value 0 (i.e., white). Therefore, we will sometimes denote X by X_1 and X^c by X_0.

Let α-adjacency denote the ordinary adjacency relation, where $\alpha \in \{6, 18, 26\}$. We could also use other adjacency relations, e.g., 14-adjacency, which is defined for 3D binary pictures in Gordon and Udupa [7]. We say that two points $p, q \in \mathbb{Z}^3$ are *α-adjacent* in digital picture $(\mathbb{Z}^3, \lambda)$ if p and q are α-adjacent and p and q have the same color, i.e., $\lambda(p) = \lambda(q)$. Similarly, we can define *α-paths* and *α-components*.

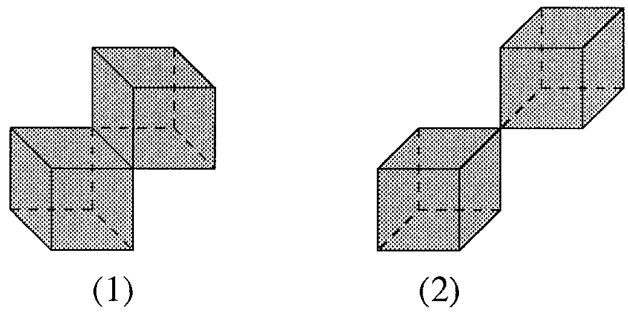

FIG. 3. A digital picture $(\mathbb{Z}^3, X)$ is well-composed iff the critical configurations of cubes (1) and (2) (modulo reflections and rotations) do not occur in $\mathrm{CA}(X_\kappa)$ for $\kappa = 0, 1$.

Recall that a subset X of $\mathbb{R}^3$ is a *2D manifold* if each point in X has a neighborhood homeomorphic to $\mathbb{R}^2$.

DEFINITION. We will call a 3D digital picture $(\mathbb{Z}^3, X)$ *well-composed* if $\mathrm{bdCA}(X)$ is a 2D manifold.

Since $\mathrm{bdCA}(X) = \mathrm{bdCA}(X^c)$, $(\mathbb{Z}^3, X)$ is well-composed iff $(\mathbb{Z}^3, X^c)$ is well-composed. This definition can be visualized by Proposition 2.1, which shows the equivalence of this definition to two simple local conditions on cubes in the continuous analog.

PROPOSITION 2.1. *A digital picture $(\mathbb{Z}^3, X)$ is well-composed iff the following critical configurations of cubes* (1) *and* (2) (*modulo reflections and rotations*) *do not occur in* $\mathrm{CA}(X_\kappa)$ *for* $\kappa = 0, 1$ (*see Fig.* 3), *where* $X_1 = X$ *and* $X_0 = X^c$:

(1) *Four cubes share an edge and exactly two of them which do not share a face are contained in* $\mathrm{CA}(X_\kappa)$ *and the other two are not contained in* $\mathrm{CA}(X_\kappa)$.

(2) *Eight cubes share a corner point and exactly two of them which are corner-adjacent are contained in* $\mathrm{CA}(X_\kappa)$ *while the other six are not.*

Proof. "$\Rightarrow$" Evidently, if $(\mathbb{Z}^3, X)$ is well-composed, then configurations (1) and (2) do not occur in $\mathrm{CA}(X_\kappa)$, since any interior point of the common edge of the two cubes in $\mathrm{CA}(X_\kappa)$ in (1) and the common vertex of the two cubes in (2) do not have neighborhoods homeomorphic to $\mathbb{R}^2$ for $\kappa = 0, 1$ (see Fig. 3).

"$\Leftarrow$" Assume now that configurations (1) and (2) do not occur in $\mathrm{CA}(X)$ where $X = X_\kappa$ for $\kappa = 0, 1$. We recall that the face boundary $\mathrm{bdCA}(X)$ is the union of the set of closed faces each of which is the common face of a cube in $\mathrm{CA}(X)$ and a cube in $\mathrm{CA}(X^c)$. Clearly, if a point $x \in \mathrm{bdCA}(X)$ lies in the interior of some square contained in the boundary $\mathrm{bdCA}(X)$, than x has a neighborhood homeomorphic to $\mathbb{R}^2$.

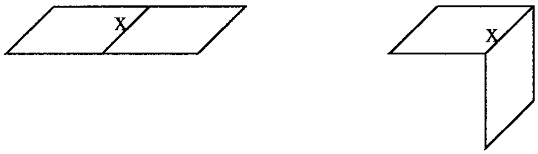

FIG. 4. In the continuous analog of a 3D well-composed picture, exactly two boundary faces can have a common edge.

Now we consider the case in which $x \in \mathrm{bdCA}(X)$ lies in the interior of some line segment that is an edge contained in the boundary $\mathrm{bdCA}(X)$. Since configuration (1) does not occur in $\mathrm{CA}(X)$, boundary faces of $\mathrm{CA}(X)$ that contain point x can have only one of the two configurations shown in Fig. 4 (modulo rotations and reflections). Thus, x has a neighborhood homeomorphic to $\mathbb{R}^2$.

It remains to consider the case in which $x \in \mathrm{bdCA}(X)$ is a corner point of $\mathrm{bdCA}(X)$. In this case eight cubes share x as their common corner point; some of them are contained in $\mathrm{CA}(X)$ and some are not. By simple analysis of all possible configurations of the eight cubes, we will obtain that boundary faces of $\mathrm{CA}(X)$ that contain point x can have only the configurations shown in Fig. 5 (modulo rotations and reflections). This implies that x has a neighborhood homeomorphic to $\mathbb{R}^2$.

We start this analysis with one cube $q \subset \mathbb{R}^3$ whose corner point is x such that $q = \mathrm{CA}(p)$ for some point $p \in X \subset \mathbb{Z}^3$. If all other cubes whose corner point is x are contained in $\mathrm{CA}(X^c)$, then boundary faces of q that contain x form the configuration in Fig. 5a. If there is one more cube r contained in $\mathrm{CA}(X)$ that shares x with q, r must share a face with q, since configurations (1) and (2) are not allowed. Thus, boundary faces of $q \cup r$ that contain x form configuration in Fig. 5b. By similar arguments, if we add a third cube, we only obtain configuration 5c of boundary faces. If we add a forth cube, we obtain one of the configurations 5d, 5e, or 5f.

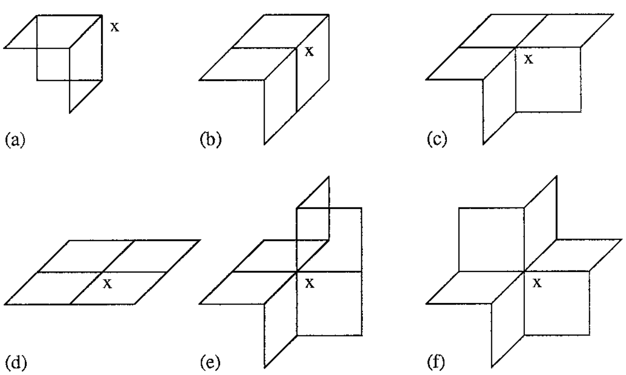

FIG. 5. In the continuous analog of a 3D well-composed picture, only these configurations of boundary faces can occur around a corner point of the object boundary.

Adding a fifth cube will transform the configurations 5d, 5e, or 5f of boundary faces to configuration 5c, which is now viewed as having five cubes in $\mathrm{CA}(X)$. Adding a sixth cube will transform configuration 5c of boundary faces (of five cubes) to configuration 5b, which is now viewed as having six cubes in $\mathrm{CA}(X)$. Adding a seventh cube will yield configuration 5a of boundary faces of seven cubes in $\mathrm{CA}(X)$. Thus, we have shown that boundary faces of $\mathrm{CA}(X)$ that contain point x can only have the six configurations in Fig. 5 (modulo rotations and reflections). ∎

Observe that the six face neighborhoods of a corner point shown in Fig. 5 are exactly the same as shown in Chen and Zhang [4] and in Francon [6]. In Artzy *et al.* [2] the digital 3D sets that do not contain configuration (1) in Fig. 3 (modulo reflections and rotations) are defined as solid. However, configuration (2) can occur in a solid set. As a simple consequence of Proposition 2.1, we obtain the following equivalent definition of well-composedness:

PROPOSITION 2.2. *A digital picture* $(\mathbb{Z}^3, X)$ *is well-composed iff for any corner point* $x \in \mathrm{bdCA}(X)$, *the boundary faces of* $\mathrm{CA}(X)$ *that contain x have one of the six configurations shown in Fig.* 5 (*modulo reflections and rotations*).

Proof. "$\Rightarrow$" If $(\mathbb{Z}^3, X)$ is well-composed, the configurations (1) and (2) (modulo reflections and rotations) do not occur in $\mathrm{CA}(X_\kappa)$ for $\kappa = 0, 1$, by Proposition 2.1. By the second part of the proof of Proposition 2.1, we obtain that if configurations (1) and (2) do not occur in $\mathrm{CA}(X_\kappa)$, then the boundary faces of $\mathrm{CA}(X)$ that contain point x can have only the configurations shown in Fig. 5.

"$\Leftarrow$" Since every point $y \in \mathrm{bdCA}(X_\kappa)$ is an interior point (in the 2D sense) of one of the configurations of faces shown in Fig. 5, y has a neighborhood homeomorphic to $\mathbb{R}^2$. Thus, $(\mathbb{Z}^3, X)$ is well-composed. ∎

Observe that there is only one connectedness relation on faces contained in the boundary of the continuous analog $\mathrm{CA}(X)$ of a well-composed picture $(\mathbb{Z}^3, X)$: A set of boundary faces S is a corner-connected component of $\mathrm{bdCA}(X)$ iff S is an edge-connected component of $\mathrm{bdCA}(X)$.

Since every boundary $\mathrm{bdCA}(X)$ is a finite union of some set of closed faces S, i.e., $\mathrm{bdCA}(X) = \bigcup S$, the statement that $\mathrm{bdCA}(X)$ is a *simple closed surface* means here that $\mathrm{bdCA}(X)$ is a connected 2D manifold in $\mathbb{R}^3$. Hence, we obtain the following proposition as a direct consequence of the definition of a well-composed picture.

PROPOSITION 2.3. *A digital picture* $(\mathbb{Z}^3, X)$ *is well-composed iff every component of* $\mathrm{bdCA}(X)$ *is a simple closed surface.*

Observe also that a set of $X \subseteq \mathbb{Z}^3$ is well-composed iff $\mathrm{CA}(X)$ is a bordered 3D manifold, where a closed set $A \subseteq \mathbb{R}^3$ is a *bordered 3D manifold* if every point in A has a neighborhood homeomorphic to a relatively open subset of

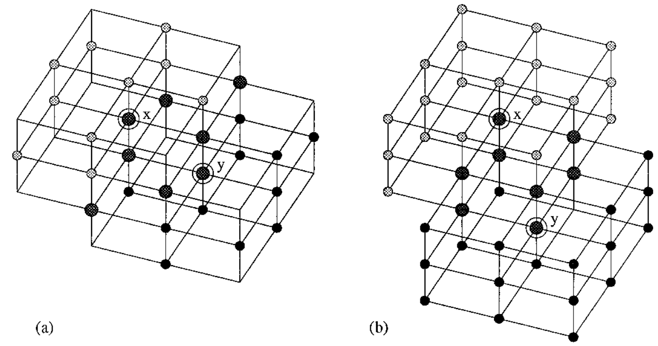

FIG. 6. The slightly larger black balls illustrate in (a) the intersection $\mathcal{N}_{18}(x) \cap \mathcal{N}_{18}(y)$ of two 18- but not 6-adjacent points x and y, and in (b) the intersection $\mathcal{N}(x) \cap \mathcal{N}(y)$ of two 26- but not 18-adjacent points x and y.

a closed half-space in $\mathbb{R}^3$. Now we give a "digital characterization" (using only points in $\mathbb{Z}^3$) of well-composed pictures.

PROPOSITION 2.4. *A 3D digital picture* $(\mathbb{Z}^3, X)$ *is well-composed iff the following conditions hold for* $\kappa = 0, 1$ (*where* $X_1 = X$ *and* $X_0 = X^c$):

(C1) *for every two 18-adjacent points* x, y *in* X_κ, *there is a 6-path joining* x *to* y *in* $\mathcal{N}_{18}(x) \cap \mathcal{N}_{18}(y) \cap X_\kappa$ *and*

(C2) *for every two 26-adjacent points* x, y *in* X_κ, *there is a 6-path joining* x *to* y *in* $\mathcal{N}(x) \cap \mathcal{N}(y) \cap X_\kappa$.

Proof. Let $X = X_\kappa$, where $\kappa = 0, 1$. We show first that the negation of condition (C1) is equivalent to the fact that configuration (1) (Fig. 3) occurs in CA(X).

If configuration (1) occurs in CA(X), then there exists four distinct points $x, y \in X$ and $a, b \notin X$ such that CA(x), CA(y), CA(a), CA(b) share an edge. Then $x, y \in X$ are 18- but not 6-adjacent in X. Figure 6a shows the intersection $\mathcal{N}_{18}(x) \cap \mathcal{N}_{18}(y)$ of two 18- but not 6-adjacent points x and y. It is easily seen that there is no 6-path joining x to y in $\mathcal{N}_{18}(x) \cap \mathcal{N}_{18}(y) \cap X$.

Conversely, if there exists two 18-adjacent points x, y in X such that there is no 6-path joining x to y in $\mathcal{N}_{18}(x) \cap \mathcal{N}_{18}(y) \cap X$, then x and y are 18- but not 6-adjacent. Hence cubes CA(x) and CA(y) share an edge, and the other two cubes that share the same edge are not contained in CA(X). Therefore, the configuration (1) (Fig. 3) occurs in CA(X), and, by Proposition 2.1, $(\mathbb{Z}^3, X)$ is not well-composed.

Now we show that if configuration (2) (Fig. 3) occurs in CA(X), then condition (C2) does not hold. Let $x, y \in X$ be such that CA(x) and CA(y) form configuration (2). Then $x, y \in X$ are 26- but not 18-adjacent in X. Figure 6b shows the intersection $\mathcal{N}(x) \cap \mathcal{N}(y)$ of two 26- but not 18-adjacent points x and y. It is easily seen that the other six points in $\mathcal{N}(x) \cap \mathcal{N}(y)$ do not belong to X. Therefore, there is no 6-path joining x to y in $\mathcal{N}(x) \cap \mathcal{N}(y) \cap X_\kappa$.

Finally, we assume the negation of condition (C2). Let x, y in X be two 26-adjacent points such that there is no 6-path joining x to y in $\mathcal{N}(x) \cap \mathcal{N}(y) \cap X$. This implies that configuration (2) or configuration (1) occurs in CA(X). ■

The following proposition implies that there is only one kind of connected components in a well-composed picture, since 26-, 18-, and 6-connected components are equal.

PROPOSITION 2.5. *Let* $(\mathbb{Z}^3, X)$ *be a well-composed picture. Then each 26-component of* X_κ *is a 6-component of* X_κ *and each 18-component of* X_κ *is a 6-component of* X_κ, *where* $\kappa = 0, 1$.

Proof. Let $x = x_1, x_2, \ldots, x_n = y$ be a 26-path joining x to y in X_κ. By condition (C2) in Proposition 2.4, for any two 26-neighbors x_i, x_{i+1}, $i = 1, \ldots, n-1$, there is a 6-path joining x_i to x_{i+1} in X_κ. Thus, there exists a 6-path joining x to y in X_κ. The argument for 18-components is similar. ■

3. JORDAN–BROUWER SEPARATION THEOREM

An important motivation for introducing 3D well-composed pictures is the following digital version of the Jordan–Brouwer separation theorem. We recall that in a digital picture $(\mathbb{Z}^3, X)$ either $X_1 = X$ or its complement $X_0 = X^c$ is finite and nonempty.

THEOREM 3.1. *If a 3D digital picture* $(\mathbb{Z}^3, X)$ *is well-composed, then for every connected component* S *of* bdCA(X), $\mathbb{R}^3 \backslash S$ *has precisely two connected components of which* S *is the common boundary.*

Proof. The proof of this theorem folllows directly from Theorem 3.2, which is stated at the end of this section. It is sufficient to observe that by Proposition 2.2, a connected component of bdCA(X) is a strongly connected polyhedral surface without boundary, which we define below. ■

Note that if a digital picture is not well-composed, Theorem 3.1 does not hold, for example, if X is a two-point digital set such that CA(X) is as shown in Fig. 3.

Now we define polyhedral surfaces in $\mathbb{R}^3$. They were used in Kong and Roscoe [10] to prove 3D digital analogs of the Jordan Curve Theorem. Let $n \geq 0$ and let $\{T_i : 0 \leq i \leq n\}$ be a set of closed triangles in $\mathbb{R}^3$. The set $\bigcup\{T_i : 0 \leq i \leq n\}$ is called a *polyhedral surface* if the following conditions both hold:

(i) If $i \neq j$, then $T_i \cap T_j$ is either a side of both T_i and T_j or a corner of both T_i and T_j or the empty set.

(ii) Each side of a triangle T_i is a side of at most one other triangle.

The (1D) *boundary of a polyhedral surface* $S = \bigcup\{T_i : 0 \le i \le n\}$ is defined as $\bigcup\{s : s$ is a side of exactly one $T_i\}$. Observe that this definition produces the same boundary of S for every dissection of S into triangles fulfilling (i) and (ii). We say that S is a *polyhedral surface without boundary* if the boundary of S is the empty set. A polyhedral surface S is *strongly connected* if for any finite set of points $F \subseteq S$, the set $S \backslash F$ is polygonally connected, where the definition of a polygonally connected set is the following:

If u and v are two distinct points in $\mathbb{R}^3$, then uv denotes the straight line segment joining u to v. Suppose $n \ge 0$ and $\{x_i : 0 \le i \le n\}$ is a set of distinct points in $\mathbb{R}^3$ such that whenever $i \ne j$, $x_i x_{i+1} \cap x_j x_{j+1} = \{x_i, x_{i+1}\} \cap \{x_j, x_{j+1}\}$, then $\mathrm{arc}(x_0, x_n) = \{x_i x_{i+1} : 0 \le i < n\}$ is a *simple polygonal arc* joining x_0 to x_n. We call a subset S of $\mathbb{R}^3$ *polygonally connected* if any two points in S can be joined by a simple polygonal arc contained in S.

Now we can state the Jordan–Brouwer separation theorem for a strongly connected polyhedral surface without boundary. This theorem is a very important result of combinatorial topology (e.g., see Aleksandrov [1]). It was applied in Kong and Roscoe [10] to establish separation theorems for digital surfaces:

THEOREM 3.2. *If S is a strongly connected polyhedral surface without boundary then $\mathbb{R}^3 \backslash S$ has precisely two components, and one of the components is bounded. S is the boundary of each component.*

Our proof is Theorem 3.1 is based on the Jordan–Brouwer separation theorem stated in Theorem 3.2, which is a powerful tool of combinatorial topology. Therefore, it seems to be an interesting question whether it is possible to derive a simple proof of Theorem 3.1 directly in discrete topology.

4. PROPERTIES OF BOUNDARY FACES

Recall that we interpret $\mathbb{Z}^3$ as a set of points with integer coordinates in the space $\mathbb{R}^3$, $\mathscr{C}$ is a set of closed unit upright cubes which are centered at points of $\mathbb{Z}^3$, and $\mathscr{F}$ is a set of closed faces of cubes in $\mathscr{C}$; i.e., each $f \in \mathscr{F}$ is a unit closed square in $\mathbb{R}^3$ parallel to one of the coordinate planes. Note that $\mathscr{C} = \{\mathrm{CA}(p) : p \in \mathbb{Z}^3\}$ and $\mathscr{F} = \{\mathrm{CA}((p, q)) : p, q \in \mathbb{Z}^3$ and p is 6 adjacent to $q\}$. We also recall that the function $\mathrm{Dig}_\in : \mathscr{P}(\mathbb{R}^3) \to \mathscr{P}(\mathbb{Z}^3)$ is defined by $\mathrm{Dig}_\in(Y) = \{p \in \mathbb{Z}^3 : p \in Y\}$. We begin this section with a theorem relating well-composed pictures to simple closed surfaces composed of faces in $\mathscr{F}$.

THEOREM 4.1. *Let $S \subset \mathscr{F}$ be a finite and nonempty set of faces in $\mathbb{R}^3$. $\bigcup S$ is a simple closed surface i.e., $\bigcup S$ is a connected and compact 2D manifold in $\mathbb{R}^3$) iff $\mathbb{R}^3 \backslash \bigcup S$ has precisely two components X_1 and X_2, $\bigcup S$ is the common boundary of X_1 and X_2, and the binary digital picture $(\mathbb{Z}^3, \mathrm{Dig}_\in(X_1))$ is well-composed.*

The proof of this theorem will be given below. Observe that the implication "$\Leftarrow$" in Theorem 4.1 would not be true if the set $\mathrm{Dig}_\in(X_1)$ were not well-composed. Let $S = \mathrm{bdCA}(D)$, where D is a digital set of 1's in the following $2 \times 2 \times 2$ configuration (on a background of 0's).

$$\begin{matrix} 1 & 0 & \quad & 1 & 1 \\ 1 & 1 & \quad & 0 & 1 \end{matrix}$$

Then $\mathbb{R}^3 \backslash S$ has precisely two components, but S is not a simple closed surface, since the common corner of the six black (i.e., 1-) voxels does not have a neighborhood homeomorphic to $\mathbb{R}^2$.

To better understand the equivalence in Theorem 4.1, we consider again the six simple local configuration of faces shown in Fig. 5.

THEOREM 4.2. *If $S \subset \mathscr{F}$ is a finite and nonempty set of faces in $\mathbb{R}^3$, then the following conditions are equivalent:*

(i) *$\bigcup S$ is a simple closed surface (i.e., $\bigcup S$ is a connected and compact 2D manifold in $\mathbb{R}^3$);*

(ii) *S is corner-connected and for every corner point $x \in \bigcup S$, the boundary faces of S that contain x as their corner point have one of the six configurations shown in Fig. 5 (modulo reflections and rotations).*

Proof. "(i) $\Rightarrow$ (ii)" Since $\bigcup S$ is a simple closed surface, each point $x \in \bigcup S$ has a neighborhood homeomorphic to $\mathbb{R}^2$. Thus, in particular, each corner point x of a face in S has a neighborhood homeomorphic to $\mathbb{R}^2$. By simple case checking (similar to one in the second part of the proof of Proposition 2.1), it can be shown that Fig. 5 shows all possible configurations (modulo rotations and reflections) of faces in $\mathscr{F}$ that share a common corner point x such that x has a neighborhood homeomorphic to $\mathbb{R}^2$. Now since $\bigcup S$ is connected, the set of faces S must be corner-connected. Thus, we obtain (i) $\Rightarrow$ (ii).

"(ii) $\Rightarrow$ (i)" We assume (ii). Then every point in the 2D interior of a face in S, clearly has a neighborhood homeomorphic to $\mathbb{R}^2$. Since every edge belongs to exactly two faces in S, every point of an edge (except the two corner points) has a neighborhood homeomorphic to $\mathbb{R}^2$. Since for every corner point x of a face in S, the set of faces sharing x has one of the six configurations of faces shown in Fig. 5, x has a neighborhood homeomorphic to $\mathbb{R}^2$. Thus, $\bigcup S$ is a 2D manifold. $\bigcup S$ is a connected subset of $\mathbb{R}^3$, since S is corner-connected. Since $\bigcup S$ is a finite union of closed squares in $\mathbb{R}^3$, $\bigcup S$ is compact. Therefore, $\bigcup S$ is a simple closed surface. ■

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. "$\Rightarrow$" Let $\bigcup S$ be a simple closed surface. Then S satisfies condition (ii) of Theorem 4.2. Consequently, $\bigcup S$ is a strongly connected polyhedral surface without boundary. By Theorem 3.2, $\mathbb{R}^3 \backslash \bigcup S$ has precisely two components X_1 and X_2, and $\bigcup S$ is the common boundary of X_1 and X_2. It remains to show that the digital picture $(\mathbb{Z}^3, \mathrm{Dig}_{\in}(X_1))$ is well-composed.

Note that $\mathrm{Dig}_{\in}(X_1)$ is the set of black points and $\mathrm{Dig}_{\in}(X_2)$ is the set of white points in $(\mathbb{Z}^3, \mathrm{Dig}_{\in}(X_1))$. Since $X_i \cup \bigcup S = \mathrm{CA}(\mathrm{Dig}_{\in}(X_i))$, we have, for $i = 1, 2$ $\bigcup S = \mathrm{bd}(\mathrm{CA}(\mathrm{Dig}_{\in}(X_i)))$. Thus, the boundaries of the sets of black and white points are 2D manifolds. We obtain that $(\mathbb{Z}^3, \mathrm{Dig}_{\in}(X_1))$ is well-composed.

"$\Leftarrow$" Since $(\mathbb{Z}^3, \mathrm{Dig}_{\in}(X_1))$ is a well-composed picture, $\mathrm{bd}(\mathrm{CA}(\mathrm{Dig}_{\in}(X_1)))$ is a 2D manifold. Since the closed set $X_1 \cup \bigcup S$ is a union of some cubes in $\mathscr{C}$, we obtain $X_1 \cup \bigcup S = \mathrm{CA}(\mathrm{Dig}_{\in}(X_1))$. Hence $\bigcup S = \mathrm{bd}(\mathrm{CA}(\mathrm{Dig}_{\in}(X_1)))$, which means that $\bigcup S$ is a 2D manifold in $\mathbb{R}^3$.

Since $\bigcup S$ is a finite union of closed squares in $\mathbb{R}^3$, it is compact. It remains to show that $\bigcup S$ is connected. If $\bigcup S$ were not connected, then there would be more than two components of $\mathbb{R}^3 \backslash \bigcup S$, since every connected component of $\bigcup S$ would be a strongly connected polyhedral surface without boundary, and, therefore, it would satisfy Theorem 3.2. ∎

5. SURFACES IN THE SENSE OF MORGENTHALER AND ROSENFELD

In our approach we treat the surface of a digital object $X \subseteq Z^3$ as described in Herman [9], i.e., as the set of pairs of 6-adjacent points (p, q), where $p \in X$ and $q \in X^c$. In this way, these pairs correspond to faces of cubes in $\mathrm{CA}(X)$ that are contained in $\mathrm{bdCA}(X)$. In computer vision literature, a surface of a 3D digital object is also interpreted as being composed of image points. This approach is taken in Morgenthaler and Rosenfeld [14], where digital simple closed surfaces are defined with a goal that they have the Jordan separability property; i.e., if $S \subseteq \mathbb{Z}^3$ is a simple closed surface, then $\mathbb{Z}^3 \backslash S$ has precisely two components. We will call a digital simple closed surface in the sense of Morgenthaler and Rosenfeld [14] a M–R surface, where a 6-connected digital set $S \subseteq \mathbb{Z}^3$ in a digital picture $(\mathbb{Z}^3, S, 6, 26)$ is defined to be a M–R surface if the following three conditions hold for every point $p \in S$ (recall that in a digital picture $(\mathbb{Z}^3, S, 6, 26)$, 6-adjacency is considered for points in X and 26-adjacency for points in X^c):

1. $S \cap \mathcal{N}(p)$ has exactly one 6-component 6-adjacent to p;
2. $S^c \cap \mathcal{N}(p)$ has exactly two 26-components $C_1(p)$, $C_2(p)$ 26-adjacent to p;
3. If $q \in S$ and q is 6-adjacent to p, then q is 26-adjacent to both $C_1(p)$ and $C_2(p)$.

We will interpret the points of a digital picture $(\mathbb{Z}^3, X, 6, 26)$ as points of the following subset of the space $\mathbb{R}^3$:

$$\mathbb{Z}^3 + \tfrac{1}{2} = \{(k + \tfrac{1}{2}, l + \tfrac{1}{2}, m + \tfrac{1}{2}) : k, l, m \in \mathbb{Z}\}.$$

To avoid confusions, we will denote $(\mathbb{Z}^3, X, 6, 26)$ by $(\mathbb{Z}^3 + (1/2), X, 6, 26)$ in the subsequent considerations. In this way, the points of the digital picture $(\mathbb{Z}^3 + (1/2), X, 6, 26)$ are the corner points of cubes in $\mathscr{C}$ (that are centered at points of $\mathbb{Z}^3$) and also the corner points of faces in $\mathscr{F}$. Hence the boundary faces of pictures $(\mathbb{Z}^3 + (1/2), X, 6, 26)$ and pictures $(\mathbb{Z}^3, X)$ are the same. The continuous analog of surfaces made of image points is defined in Kong and Roscoe [10]. Based on this definition, a *Kong's continuous analog* $\mathrm{KA}(S)$ of a M–R surface $S \subseteq \mathbb{Z}^3$ (treated as a digital picture $(\mathbb{Z}^3 + (1/2), S, 6, 26)$) is the union of all faces $f \in \mathscr{F}$ such that all four corner points of f are in S. By the results in Chen and Zhang [4] (Theorems 2.1 and 4.1), for every point x in a M–R surface $S \subseteq \mathbb{Z}^3 + (1/2)$, the faces in $\mathrm{KA}(S)$ that contain x (as their corner point) have one of the six configurations shown in Fig. 5 (modulo reflections and rotations). By Theorem 4.2 ((ii) $\Rightarrow$ (i)), we obtain that $\mathrm{KA}(S)$ of a M–R surface S is a simple closed surface in $\mathbb{R}^3$.

From Theorem 4.1, it follows that $\mathbb{R}^3 \backslash KA(S)$ has precisely two components X_1 and X_2, $\mathrm{KA}(S)$ is the common boundary of X_1 and X_2, and the binary digital picture $(\mathbb{Z}^3, \mathrm{Dig}_{\in}(X_1))$ is well-composed. Consequently, we obtain $\mathrm{KA}(S) = \mathrm{bdCA}(\mathrm{Dig}_{\in}(X_1))$. Thus, every M–R surface $(\mathbb{Z}^3, + (1/2), S, 6, 26)$ can be interpreted as the boundary surface of the well-composed digital picture $(\mathbb{Z}^3, \mathrm{Dig}_{\in}(X_1))$.

However, it is not the case that for every boundary surface $S = \mathrm{bdCA}(X)$ in a binary well-composed digital picture $(\mathbb{Z}^3, X)$, the digital set $(\mathbb{Z}^3 + (1/2), \mathrm{Dig}_{\in}(S), 6, 26)$ is a M–R surface. The reason is that although $S \subset \mathbb{R}^3$ is simple closed surface, the region surrounded by $\mathrm{Dig}_{\in}(S)$ can contain none of the points in $\mathbb{Z}^3 + (1/2)$. For example, the digital image $(\mathbb{Z}^3, p)$ with a single black point p is well-composed and $S = \mathrm{bdCA}(p)$ is the boundary of a unit cube centered at p. However, the digital set $\mathrm{Dig}_{\in}(S) \subset \mathbb{Z}^3 + (1/2)$, which consists of the eight corner points of the cube $\mathrm{CA}(p)$, does not surround any point in $\mathbb{Z}^3 + (1/2)$.

6. CONNECTED COMPONENTS IN 3D WELL-COMPOSED PICTURES

For a 2D digital binary picture $(\mathbb{Z}^2, X)$, a set of black points X can be identified with the union of closed unit squares centered at points of X, which we denote $\mathrm{CA}(X)$. We assume that either X or its complement X^c is finite and nonempty. The boundary $\mathrm{bdCA}(X)$ of a 2D set X is

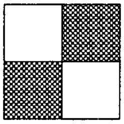

FIG. 7. The continuous analog of a 2D well-composed picture does not contain this critical configuration and its 90° rotation.

the union of the set of unit line segments each of which is the common edge of a square in $\mathrm{CA}(X)$ and a square in $\mathrm{CA}(X^c)$. Observe that there is only one kind of adjacency for line segments contained in $\mathrm{bdCA}(X)$: two segments are adjacent if they have an endpoint in common. Hence, there is only one kind of connectedness for $\mathrm{bdCA}(X)$. The unit line segments contained in $\mathrm{bdCA}(X)$ correspond to pairs of 4-adjacent points (p, q) such that $p \in X$ and $q \notin X$.

A 2D binary digital picture $(\mathbb{Z}^2, X)$ is well-composed iff the critical configuration shown in Fig. 7 (and its 90° rotation) does not occur in $\mathrm{CA}(X)$ and $\mathrm{CA}(X^c)$ (Latecki *et al.* [12]). For 2D well-composed pictures, the following theorem can be easily proven:

THEOREM 6.1. *A digital picture* $(\mathbb{Z}^2, X)$ *is well-composed iff* $\mathrm{bdCA}(X)$ *is a compact 1D manifold* (*each point in* $\mathrm{bd}X$ *has a neighborhood homeomorphic to* $\mathbb{R}$).

Rosenfeld and Kong [17] proved the following theorem for 2D digital pictures:

THEOREM 6.2. *For every finite and nonempty set* $X \subset \mathbb{Z}^2$, *the boundary* $\mathrm{bdCA}(X)$ *is a simple closed curve* (*i.e.*, $\mathrm{bdCA}(X)$ *is connected and each line segment in* $\mathrm{bdCA}(X)$ *is adjacent to exactly two others*) *iff* X *and* X^c *are both 4-connected*.

As shown in [17], an analogous theorem does not hold in 3D: Let X be a set of 1's in the following $2 \times 2 \times 2$ configuration (on a background of 0's).

$$\begin{matrix} 1 & 1 & \quad & 0 & 1 \\ 1 & 1 & \quad & 1 & 0 \end{matrix}$$

Then X and X^c are both 6-connected, but $\mathrm{bdCA}(X)$ is not a simple closed surface. However, the inverse implication is proved in [17], Proposition 9:

THEOREM 6.3. *If the boundary* $\mathrm{bdCA}(X)$ *of a set* $X \subset \mathbb{Z}^3$ *is a simple closed surface, then* X *and* X^c *are both 6-connected.*

Using the concept of well-composedness, we can generalize Theorem 6.2 to three dimensions:

THEOREM 6.4. *For every finite and nonempty set* $X \subset \mathbb{Z}^3$, *the boundary* $\mathrm{bdCA}(X)$ *is a simple closed surface iff* X *and* X^c *are both 6-connected and* $(\mathbb{Z}^3, X)$ *is well-composed.*

Proof. "$\Rightarrow$:" By Theorem 6.3, we obtain that X and X^c are both 6-connected. Since a simple closed surface is in particular a 2D manifold, we obtain that $(\mathbb{Z}^3, X)$ is well-composed.

"$\Leftarrow$:" Since X and X^c are 6-connected, $\mathrm{CA}(X)$ and $\mathrm{CA}(X^c)$ are connected subsets of $\mathbb{R}^3$ and $\mathrm{bdCA}(X)$ is their common boundary. Therefore, $\mathrm{bdCA}(X)$ is also a connected subset of $\mathbb{R}^3$. Since $X \subset \mathbb{Z}^3$ is finite, $\mathrm{bdCA}(X)$ is compact. By definition, the fact that $(\mathbb{Z}^3, X)$ is well-composed implies that $\mathrm{bdCA}(X)$ is a 2D manifold. Consequently, $\mathrm{bdCA}(X)$ is a simple closed surface. ■

7. CONCLUSIONS

We showed that a number of difficult problems in 3D digital geometry become relatively simple when we restrict our attention to 3D well-composed images. If a digital picture lacks the property of being well-composed, it seems to be possible to locally "repair" the picture by adding (or subtracting) single points. Since well-composedness is a local property, it can be decided very efficiently, in parallel, whether a given digital picture has this property or not. We designed a parallel algorithm that locally "repairs" 2D pictures in one step and proved its correctness, but the 3D result is not yet established. The other possibility, which is more promising for applications, would be to impose local conditions on the segmentation process which guarantee that the obtained 3D image is well-composed.

ACKNOWLEDGMENTS

The quality of this paper has been substantially improved due to the reviewers' comments.

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