Section 7.6
Partial Orderings

Definition: Let *R* be a relation on A. Then R is a *partial order* iff *R* is

- reflexive
- antisymmetric

and

- transitive

(A, R) is called a partially ordered set or a *poset*.

Note: It is not required that two things be related under a partial order. That's the *partial* part of it.

If two objects are always related in a poset, it is called a *total order* or *linear order* or *simple order*. In this case *(A, R)* is called a *chain*.

Examples:

- $(Z \leq)$ is a poset. In this case either $a \leq b$ or $b \leq a$ so two things are always related. Hence, $\leq$ is a total order and $(Z, \leq)$ is a chain.

• If S is a set then $(P(S),\ \ )$ is a poset. It may not be the case that A B or B A. Hence, is not a total order.

• (Z^+, 'divides') is a poset which is not a chain.

Definition: Let R be a total order on A and suppose S A. An element s in S is a *least element* of S iff sRb for every b in S.

Similarly for *greatest* element.

Note: this implies that $<a, s>$ is not in R for any a unless $a = s$. (There is nothing smaller than s under the order R).

Definition: A chain (A, R) is *well-ordered* iff every subset of A has a least element.

Examples:

• $(Z, \leq)$ is a chain but not well-ordered. Z does not have least element.

• $(N, \leq)$ is well-ordered.

• $(N, \geq)$ is not well-ordered.

Lexicographic Order

Given two posets (A_1, R_1) and (A_2, R_2) we construct an *induced* partial order R on $A_1 \times A_2$:

$$< x_1, y_1 > R < x_2, y_2 > \text{ iff}$$

- $x_1 \; R_1 \; x_2$

or

- $x_1 = x_2$ and $y_1 \; R_2 \; y_2$.

Example:

Let $A_1 = A_2 = Z^+$ and $R_1 = R_2 =$ 'divides'.

Then

- $<2, 4> R <2, 8>$ since $x_1 = x_2$ and $y_1 \; R_2 \; y_2$.
- $<2, 4>$ is not related under R to $<2, 6>$ since $x_1 = x_2$ but 4 does not divide 6.
- $<2, 4> R <4, 5>$ since $x_1 \; R_1 \; x_2$. (Note that 4 is not related to 5).

This definition extends naturally to multiple Cartesian products of partially ordered sets:

$$A_1 \times A_2 \times A_3 \times \ldots \times A_n.$$

Example: Using the same definitions of A_i and R_i as above,

• < 2, 3, 4, 5> R < 2, 3, 8, 2> since $x_1 = x_2$, $y_1 = y_2$ and 4 divides 8.

• <2, 3, 4, 5> is not related to <3, 6, 8, 10> since 2 does not divide 3.

Strings

We apply this ordering to strings of symbols where there is an underlying 'alphabetical' or partial order (which is a total order in this case) as used in dictionaries.

Example:

Let $A = \{ a, b, c\}$ and suppose R is the natural alphabetical order on A:

$$a \; R \; b \text{ and } b \; R \; c.$$

Then

• If all letters agree, shorter string is related to a longer string (comes before it in the ordering).

• If two strings have the same length then the induced partial order from the alphabetical order is used:

aabc *R* abac

Hasse or Poset Diagrams

To construct a Hasse diagram:

1) Construct a digraph representation of the poset *(A, R)* so that all arcs point up (except the loops).

2) Eliminate all loops

3) Eliminate all arcs that are redundant because of transitivity

4) eliminate the arrows at the ends of arcs since everything points up.

Example:

Construct the Hasse diagram of *(P({a, b, c}),   )*.

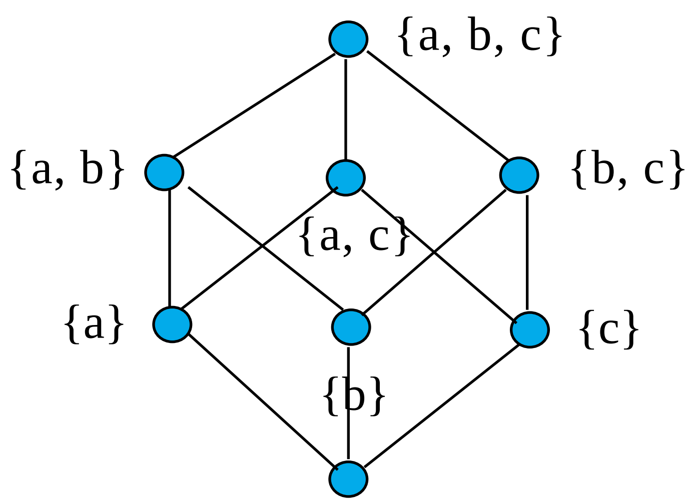
The elements of *P({a, b, c})* are

{a}, {b}, {c}

{a, b}, {a, c}, {b, c}

{a, b, c}

The digraph is



Maximal and Minimal Elements

Definition: Let *(A, R)* be a poset. Then *a* in *A* is a *minimal element* if there does not exist an element *b* in *A* such that *bRa*.

Similarly for a *maximal element*.

Note: there can be more than one minimal and maximal element in a poset.

Example: In the above Hasse diagram,   is a minimal element and $\{a, b, c\}$ is a maximal element.

Least and Greatest Elements

Definition: Let (A, R) be a poset. Then a in A is the *least element* if for every element b in A, aRb and b is the *greatest element* if for every element a in A, aRb.

Theorem: Least and greatest elements are unique.

Proof:

Assume they are not. . .

Example:

In the poset above $\{a, b, c\}$ is the greatest element.   is the least element.

Upper and Lower Bounds

Definition: Let S be a subset of A in the poset (A, R). If there exists an element a in A such that sRa for all s in S, then a is called an *upper bound*. Similarly for lower bounds.

Note: to be an upper bound you must be related to every element in the set. Similarly for lower bounds.

Example:

• In the poset above, $\{a, b, c\}$, is an upper bound for all other subsets. is a lower bound for all other subsets.

Least Upper and Greatest Lower Bounds

Definition: If a is an upper bound for S which is related to all other upper bounds then it is the *least upper bound*, denoted lub(S). Similarly for the *greatest lower bound*, glb(S).

Example:

Consider the element {a}.

Since

{a, b, c}, {a, b} {a, c} and {a}

are upper bounds and {a} is related to all of them, {a} must be the lub. It is also the glb.

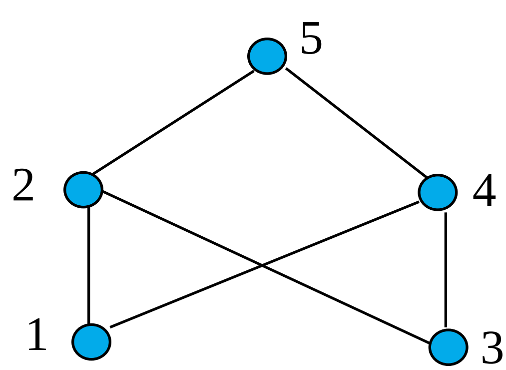
Lattices

Definition: A poset is a *lattice* if every pair of elements has a lub and a glb.

Examples:

- In the poset *(P(S),   )*, lub(*A*, *B*) = *A*   *B*. What is the glb(*A*, *B*)?

- Ex. 20, 21, 22, p. 524



Consider the elements 1 and 3.

- Upper bounds of 1 are 1, 2, 4 and 5.
- Upper bounds of 3 are 3, 2, 4 and 5.
- 2, 4 and 5 are upper bounds for the pair 1 and 3.
- There is no lub since
    - 2 is not related to 4
    - 4 is not related to 2
    - 2 and 4 are both related to 5.
- There is no glb either.

The poset is not a lattice.

Topological Sorting

We impose a total ordering R on a poset *compatible* with the partial order.

- Useful to determine an ordering of tasks.

- Useful in rendering in graphics to render objects from back to front to obscure hidden surfaces

• A painter uses a topological sort when applying paint to a canvas - he/she paints parts of the scene furthest from the view first

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Algorithm: To sort a poset (S, R).

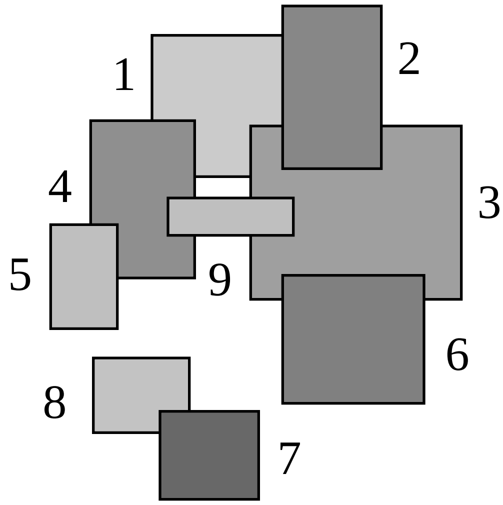
• Select a (any) minimal element and put it in the list. Delete it from S.

• Continue until all elements appear in the list (and S is void).

Example:

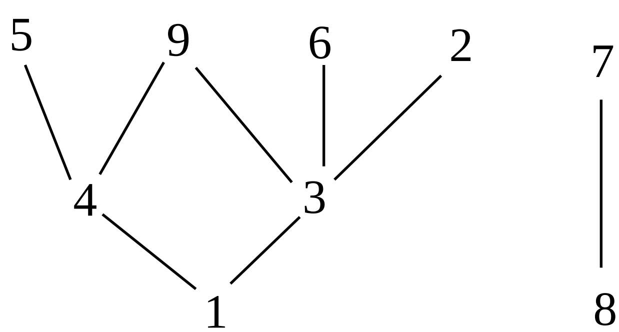
Consider the rectangles T and the relation R = "is more distant than." Then R is a partial order on the set of rectangles.

Two rectangles, T_i and T_j, are related, $T_i \; R \; T_j$, if T_i is more distant from the viewer than T_j.

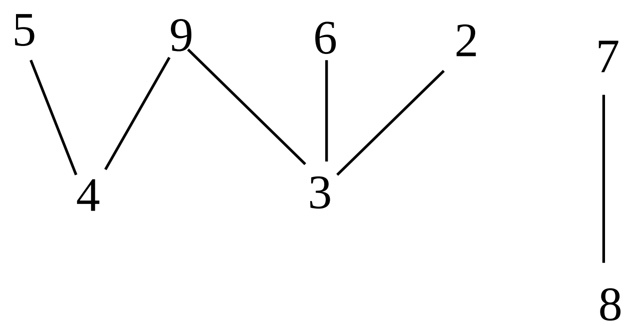


Then $1R2$, $1R4$, $1R3$, $4R9$, $4R5$, $3R2$, $3R9$, $3R6$, $8R7$.

The Hasse diagram for R is



Draw 1 (or 8) and delete 1 from the diagram to get



Now draw 4 (or 3 or 8) and delete from the diagram.
Always choose a minimal element. Any one will do.

...and so forth.

Ex. 26, p. 527, problem 59, p. 530