Section 7.5
Equivalence Relations

Now we group properties of relations together to define new types of important relations.

Definition: A relation R on a set A is an *equivalence relation* iff R is

- reflexive
- symmetric

and

- transitive

It is easy to recognize equivalence relations using digraphs.

• The subset of all elements related to a particular element forms a universal relation (contains all possible arcs) on that subset. The (sub)digraph representing the subset is called a *complete* (sub)digraph. All arcs are present.

• The number of such subsets is called the *rank* of the equivalence relation

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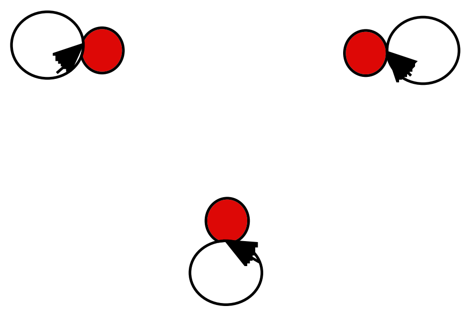
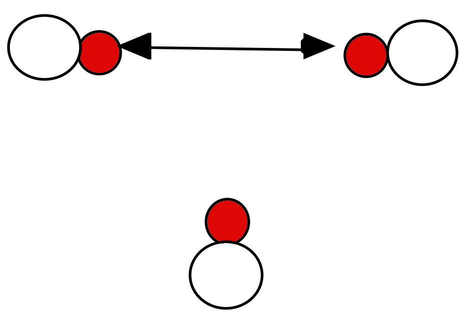
Examples:

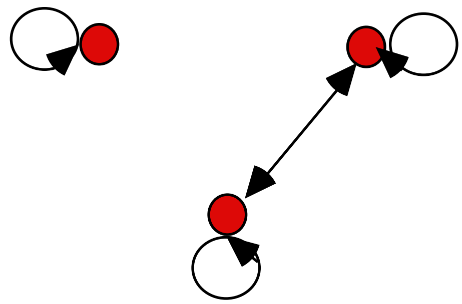
A has 3 elements:



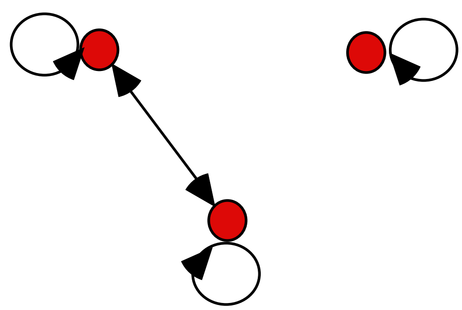
rank = 3



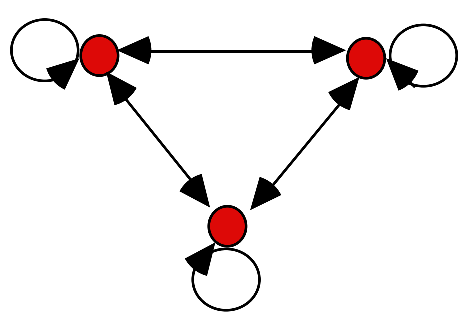
rank = 2



rank = 2



rank = 2



rank = 1

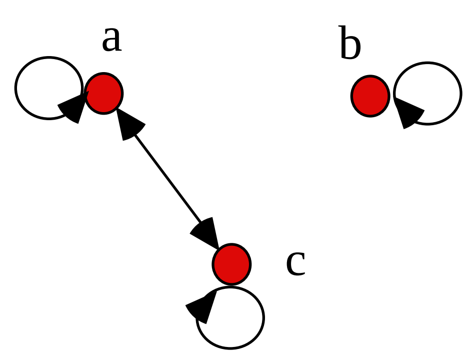
• Each of the subsets is called an *equivalence class.*

• A bracket around an element means the equivalence class in which the element lies.

$$[x] = \{y \mid \langle x, y\rangle \text{ is in } R\}$$

• The element in the bracket is called a *representative* of the equivalence class. We could have chosen any one.

Examples:



[a] = {a, c}, [c] = {a, c}, [b] = {b}.
rank = 2

An interesting counting problem:

Count the number of equivalence relations on a set A with n elements. Can you find a recurrence relation?

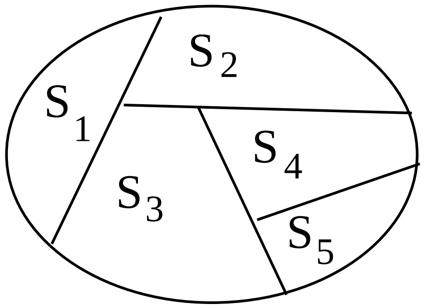
The answers are

- 1 for $n = 1$
- 2 for $n = 2$
- 5 for $n = 3$

How many for $n = 4$?

Definition: Let $S_1, S_2, \ldots, S_n$ be a collection of subsets of A. Then the collection forms a *partition* of A if the subsets are nonempty, disjoint and *exhaust* A:

- $S_i \neq$
- $S_i \cap S_j =$ if $i \neq j$
- $\bigcup S_i = A$



A

Theorem: The equivalence classes of an equivalence relation R *partition* the set A into disjoint nonempty subsets whose union is the entire set.

This partition is denoted *A/R* and called

- the *quotient set,* or
- *the partition of A induced by R,* or,
- *A modulo R.*

From: http://en.wikipedia.org/wiki/Partition_of_a_set

- The set { 1, 2, 3 } has these five partitions.
  - { {1}, {2}, {3} }, sometimes denoted by 1/2/3.
  - { {1, 2}, {3} }, sometimes denoted by 12/3.
  - { {1, 3}, {2} }, sometimes denoted by 13/2.
  - { {1}, {2, 3} }, sometimes denoted by 1/23.
  - { {1, 2, 3} }, sometimes denoted by 123.
- Note that
  - { {}, {1,3}, {2} } is not a partition if we are using axiom 1 (because it contains the empty set); otherwise it is a partition of {1, 2, 3}.
  - { {1,2}, {2, 3} } is not a partition because the element 2 is contained in more than one distinct subset.
  - { {1}, {2} } is not a partition of {1, 2, 3} because none of its blocks contains 3; however, it is a partition of {1, 2}.

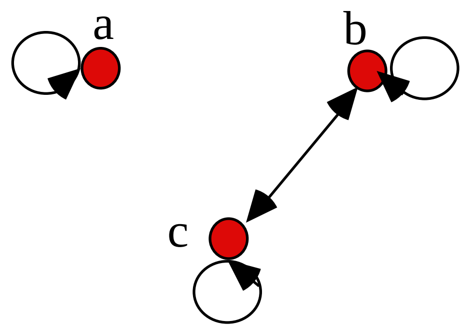
The *n*th **Bell number**, named in honor of Eric Temple Bell, is the number of partitions of a set with *n* members, or equivalently, the number of equivalence relations on it:

$$B_{n+1} = \sum_{k=0}^{n} C(n,k) B_k$$

The intuition is that we take *k* elements from *n* elements and form all their possible partitions B_k. For each of the B_k partitions, the remaining n+1-k elements from then one more set in the partition of the n+1 elements.

Examples of equivalence relations and partitions:

- Ex. 1, p. 508
- Ex. 4, p. 509
- Ex. 9, p. 512
- Problem 18, p. 514



$$A = [a] \quad [b] = [a] \quad [c] = \{a\} \quad \{b, c\}$$

rank = 2

Theorem: Let R be an equivalence relation on A. Then either

$$[a] = [b]$$

or

$$[a] \cap [b] =$$

Theorem: If R_1 and R_2 are equivalence relations on A then $R_1 \cap R_2$ is an equivalence relation on A.

Proof: It suffices to show that the intersection of

- reflexive relations is reflexive,

• symmetric relations is symmetric,

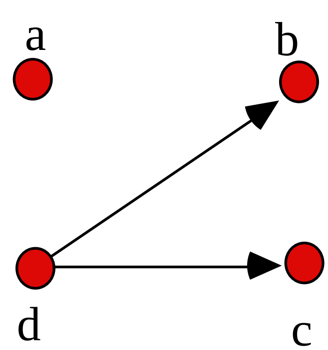
and

• transitive relations is transitive.

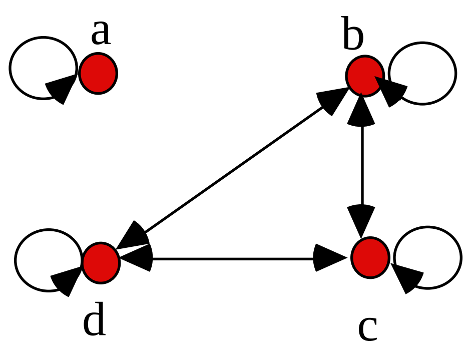
You provide the details.

Definition: Let R be a relation on A. Then the reflexive, symmetric, transitive closure of R, tsr(R), is an equivalence relation on A, called the *equivalence relation induced by R.*

Example:



R



tsr(R)
rank = 2
A = [a]   [b] = {a}   {b, c, d}

A/R = {{a}, {b, c, d}}

Theorem: tsr(R) is an equivalence relation

Proof:

We have to be careful and show that tsr(R) is still symmetric and reflexive.

• Since we only add arcs vs. deleting arcs when computing closures it must be that tsr(R) is reflexive since all loops <x, x> on the diagraph must be present when constructing r(R).

• If there is an arc <x, y> then the symmetric closure of r(R) ensures there is an arc <y, x>.

• Now argue that if we construct the transitive closure of sr(R) and we add an edge <x, z> because there is a path from x to z, then there must also exist a path from z to x (why?) and hence we also must add an edge <z, x>. Hence the transitive closure of sr(R) is symmetric.

Q. E. D.