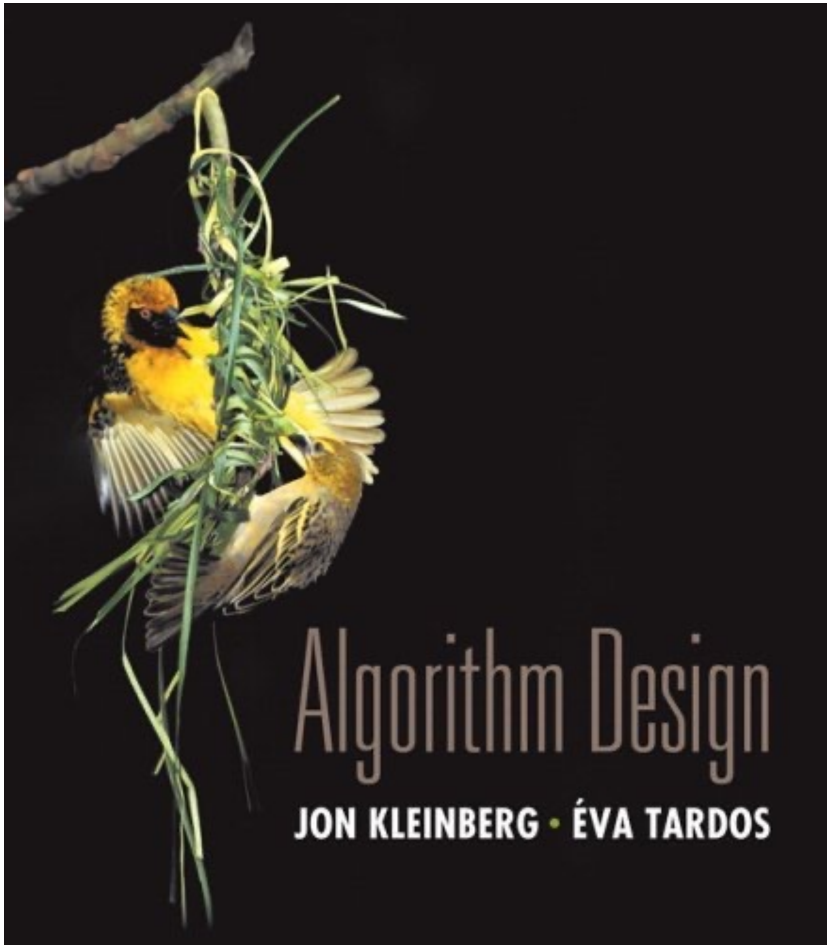
CIS 5515

Design and Analysis of Algorithms



Lectures

Lecture time

- W 5:30 pm – 8:00 pm, Tuttleman Learning Center 402
  Zoom: https://temple.zoom.us/j/7348129717 (first two classes)

Office hours

- W 3:00 pm – 5:00 pm, SERC 362

Prereq.

- CIS 5511 (Programming Techniques)

Textbook

- *Algorithm Design* by Éva Tardos and Jon Kleinberg

Course web site

- https://cis.temple.edu/~jiewu/teaching/spring_2022.html

Algorithms

Algorithm.

- [webster.com] A procedure for solving a mathematical problem (as of finding the greatest common divisor) in a finite number of steps that frequently involves repetition of an operation.

- [Knuth, TAOCP] An algorithm is a finite, definite, effective procedure, with some input and some output.

> Great algorithms are the poetry of computation. Just like verse, they can be terse, allusive, dense, and even mysterious. But once unlocked, they cast a brilliant new light on some aspect of computing. - *Francis Sullivan*

Etymology

Etymology. [Knuth, TAOCP]

- *Algorism* = process of doing arithmetic using Arabic numerals.

- True origin: Abu 'Abd Allah Muhammad ibn Musa al-Khwarizmi was a famous 9th century Persian who wrote *Kitab al-jabr wa'l-muqabala*, which evolved into today's high school algebra text.

Stamp issued by the Soviet Union on Sept. 6, 1983, to mark the 1200th birth anniversary of Al'Khowârizmî.

Scope

Design and analysis of computer algorithms

- Greedy
- Divide-and-conquer
- Dynamic programming
- Network flow
- NP and computational intractability
- Approximation algorithms
- Adversary augments for lower bounds
- …

Critical thinking, problem-solving, and complexity analysis

Two sample examples

Algorithm design: celebrity problem (Oscar 2022)

- A celebrity does not know anyone while everyone knows him/her.
- Primitive operation: select A and B and then ask A if he/she knows B.
- What is the minimum number of primitive operations needed to find a celebrity in a group of size n?

ACADEMY
OF MOTION PICTURE
ARTS AND SCIENCES

Algorithm analysis: runner-up team (Super bowl 2022)

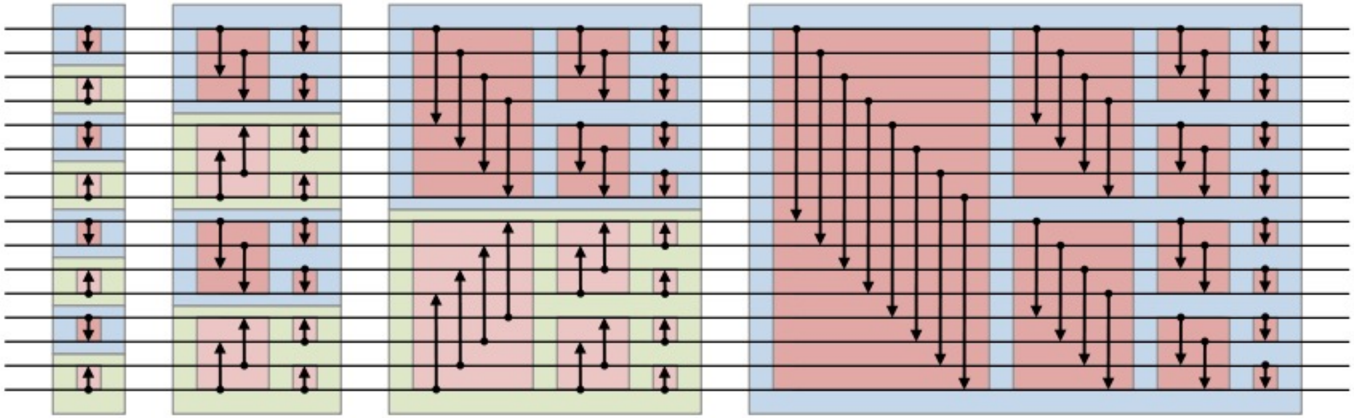
- Why no one cares about a runner-up team?
- What is the minimum number of matches needed to determine the second-best team in a league of n teams?

SUPER BOWL

Algorithmic Paradigms

- Randomized algorithm
  - Quicksort: employs a degree of randomness as part of its logic

- Parallel algorithm
  - Bitonic sort, by K. Batcher: run on many machines simultaneously ($\log^2 n$)



- Distributed algorithm (Blockchain, by S. Nakamoto)
  - Can be executed on a distributed system with no central coordinator (delay sensitive)

- Local algorithm (Connected Dominating Set on MANETs, by J. Wu)
  - Local information with global properties

Algorithmic Paradigms (cont'd)

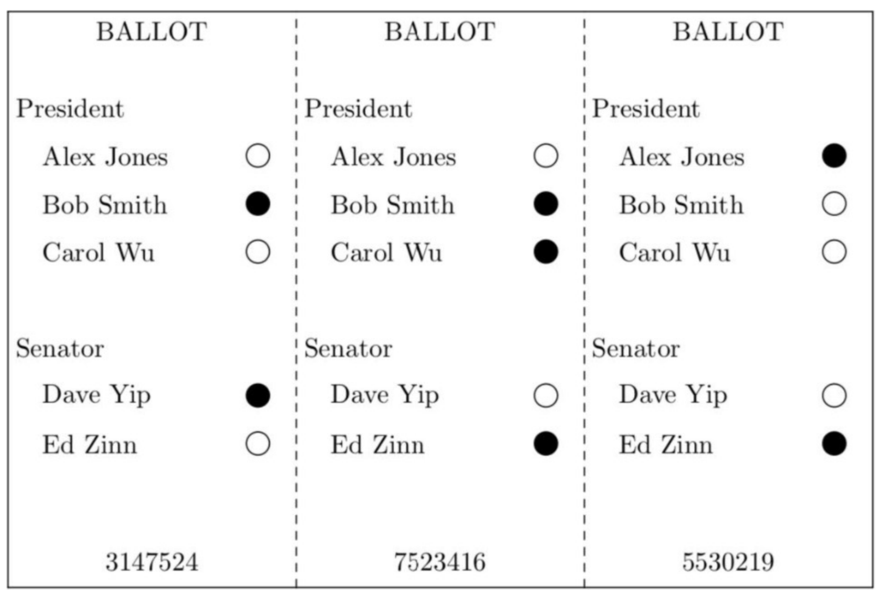
- Algorithm for security (e.g., US presidential voting)
  - Secure voting by R. Rivest

    Each voter casts 3 tickets

    For: mark exactly 2 tickets, Against: mark exactly 1 ticket

    3 tickets are separated, mixed with others, and posted in public

    Voter keeps 1 ticket for verification



Algorithmic Paradigms (cont'd)

- Privacy-preserving algorithm (The Ethical Algorithm, by M. Kearns & A. Roth)
  - Poll: do you like the former president?

    Flip a coin: head (tell the truth) and tail (tell randomly), then $\frac{3}{4}$ time is the truth
  - Algorithmic discrimination and discrimination through optimization.

- Algorithmic game theory (mechanism design to avoid bad equilibrium)
  - Two-path routing

    short path (but proportion to traffic x ($\leq$1)) and long path (fixed delay of 1)

- The Master Algorithm, by P. Domingos
  - Machine learning (ML): The algorithm to design all algorithms

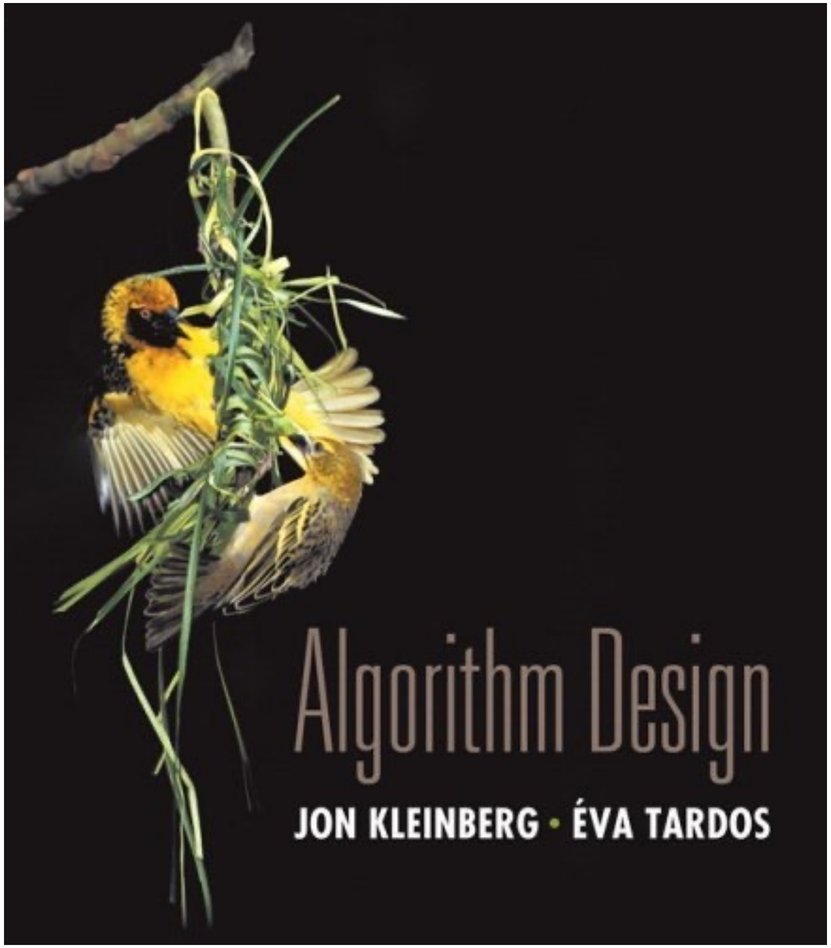
    All knowledge - past, current, and future - can be derived from data by a single, universal learning algorithm

Applications

Wide range of applications

- Caching
- Compilers
- Databases
- Scheduling
- Networking
- Data analysis
- Signal processing
- Computer graphics
- Scientific computing
- Operations research
- Artificial intelligence
- Computational biology
- . . .

We focus on algorithms and techniques that are useful in practice.



Introduction: Some Representative Problems

1.1 Dating: searching for the best mate

Searching for the best mate

Goal. Given n candidates for dating, find a "suitable" one for marriage.

- Dating in sequence.
- Accept (for marriage) or reject after each date.
- No more dating after acceptance.
- Best strategy to find a "suitable" one for marriage: optimal stopping

Phase 1: always reject the first n/e dates (e = 2.71828, natural number)
Phase 2: marry the first date better than all dates in Phase 1

1/e rule (or 37% rule). Probability of finding the best mate is 1/e

Extensions

- Unknown n
- k mates (k=1, 2, 3, 4,…)

1.1 Matching: a stable marriage

Stable Matching Problem

Goal. Given n men and n women, find a "suitable" matching.

- Participants rate members of opposite sex.
- Each man lists women in order of preference from best to worst.
- Each woman lists men in order of preference from best to worst.

	1st (favorite)	2nd	3rd (least favorite)
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	1st (favorite)	2nd	3rd (least favorite)
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Matching Problem

Perfect matching: everyone is matched monogamously.

- Each man gets exactly one woman.
- Each woman gets exactly one man.

Stability: no incentive for some pair of participants to undermine assignment by joint action.

- In matching M, an unmatched pair m-w is unstable if man m and woman w prefer each other to current partners.
- Unstable pair m-w could each improve by eloping (i.e., run away).

Stable matching: perfect matching with no unstable pairs.

Stable matching problem. Given the preference lists of n men and n women, find a stable matching if one exists.

Stable Matching Problem

Q. Is assignment X-C, Y-B, Z-A stable?

	1st (favorite)	2nd	3rd (least favorite)
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	1st (favorite)	2nd	3rd (least favorite)
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Matching Problem

Q. Is assignment X-C, Y-B, Z-A (dark gray) stable?

A. No. Bertha and Xavier will hook up.

	1st (favorite)	2nd	3rd (least favorite)
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	1st (favorite)	2nd	3rd (least favorite)
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Matching Problem

Q. Is assignment X-A, Y-B, Z-C stable?

A. Yes.

	1st (favorite)	2nd	3rd (least favorite)
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	1st (favorite)	2nd	3rd (least favorite)
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Roommate Problem

Q. Do stable matchings always exist?
A. Not obvious a priori.

is core of market nonempty?

Stable roommate problem (or marriage with one gender).

- 2n people; each person ranks others from 1 to 2n-1.
- Assign roommate pairs so that no unstable pairs.

	1st	2nd	3rd
Adam	B	C	D
Bob	C	A	D
Chris	A	B	D
Doofus	A	B	C

A-B, C-D $\Rightarrow$ B-C unstable
A-C, B-D $\Rightarrow$ A-B unstable
A-D, B-C $\Rightarrow$ A-C unstable

Observation. Stable matchings do not always exist for stable roommate problem.

Propose-And-Reject Algorithm

Propose-and-reject algorithm. [Gale-Shapley 1962] Intuitive method that guarantees to find a stable matching.

```
Initialize each person to be free.
while (some man is free and hasn't proposed to every woman) {
    Choose such a man m
    w = 1st woman on m's list to whom m has not yet proposed
    if (w is free)
        assign m and w to be engaged
    else if (w prefers m to her fiancé m')
        assign m and w to be engaged, and m' to be free
    else
        w rejects m
}
```

Proof of Correctness: Termination

Observation 1. Men propose to women in decreasing order of preference.

Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up."

Claim. Algorithm terminates after at most n^2 iterations of while loop.
Pf. Each time through the while loop a man proposes to a new woman. There are only n^2 possible proposals. ▪

	1st	2nd	3rd	4th	5th
Victor	A	B	C	D	E
Wyatt	B	C	D	A	E
Xavier	C	D	A	B	E
Yancey	D	A	B	C	E
Zeus	A	B	C	D	E

	1st	2nd	3rd	4th	5th
Amy	W	X	Y	Z	V
Bertha	X	Y	Z	V	W
Clare	Y	Z	V	W	X
Diane	Z	V	W	X	Y
Erika	V	W	X	Y	Z

$n(n-1) + 1$ proposals required

Proof of Correctness: Perfection

Claim. All men and women get matched.

Pf. (by contradiction)

- Suppose, for sake of contradiction, that Zeus is not matched upon termination of algorithm.
- Then some woman, say Amy, is not matched upon termination.
- By Observation 2, Amy was never proposed to.
- But, Zeus proposes to everyone, since he ends up unmatched. ▪

Proof of Correctness: Stability

Claim. No unstable pairs.

Pf. (by contradiction)

- Suppose A-Z is an unstable pair: each prefers each other to partner in Gale-Shapley matching S*.

- Case 1: Z never proposed to A.
  ⇒ Z prefers his GS partner to A.
  ⇒ A-Z is stable.

men propose in decreasing order of preference

- Case 2: Z proposed to A.
  ⇒ A rejected Z (right away or later)
  ⇒ A prefers her GS partner to Z. ← women only trade up
  ⇒ A-Z is stable.

- In either case A-Z is stable, a contradiction. ▪

S*
Amy-Yancey
Bertha-Zeus
. . .

Summary

Stable matching problem. Given n men and n women, and their preferences, find a stable matching if one exists.

Gale-Shapley algorithm. Guarantees to find a stable matching for any problem instance.

Q. How to implement GS algorithm efficiently?

Q. If there are multiple stable matchings, which one does GS find?

Efficient Implementation

Efficient implementation. We describe $O(n^2)$ time implementation.

Representing men and women.

- Assume men are named 1, …, n.
- Assume women are named 1', …, n'.

Engagements.

- Maintain a list of free men, e.g., in a queue.
- Maintain two arrays `wife[m]`, and `husband[w]`.
  - set entry to `0` if unmatched
  - if m matched to w then `wife[m]=w` and `husband[w]=m`

Men proposing.

- For each man, maintain a list of women, ordered by preference.
- Maintain an array `count[m]` that counts the number of proposals made by man `m`.

Efficient Implementation

Women rejecting/accepting.

- Does woman `w` prefer man `m` to man `m'`?
- For each woman, create inverse of preference list of men.
- Constant time access for each query after O(n) preprocessing.

Amy	1st	2nd	3rd	4th	5th	6th	7th	8th
Pref	8	3	7	1	4	5	6	2

Amy	1	2	3	4	5	6	7	8
Inverse	4th	8th	2nd	5th	6th	7th	3rd	1st

```
for i = 1 to n
   inverse[pref[i]] = i
```

Amy prefers man 3 to 6
since `inverse[3]` < `inverse[6]`

2 7

Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

An instance with two stable matchings.

- A-X, B-Y, C-Z.
- A-Y, B-X, C-Z.

	1st	2nd	3rd
Xavier	A	B	C
Yancey	B	A	C
Zeus	A	B	C

	1st	2nd	3rd
Amy	Y	X	Z
Bertha	X	Y	Z
Clare	X	Y	Z

Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

Def. Man m is a valid partner of woman w if there exists some stable matching in which they are matched.

Man-optimal assignment. Each man receives best valid partner.

Claim. All executions of GS yield man-optimal assignment, which is a stable matching!

- No reason a priori to believe that man-optimal assignment is perfect, let alone stable.
- Simultaneously best for each and every man.

Man Optimality

Claim. GS matching S^* is man-optimal.

Pf. (by contradiction)

- Suppose some man is paired with someone other than best partner. Men propose in decreasing order of preference $\Rightarrow$ some man is rejected by valid partner.
- Let Y be first such man, and let A be first valid woman that rejects him.
- Let S be a stable matching where A and Y are matched.
- When Y is rejected, A forms (or reaffirms) engagement with a man, say Z, whom she prefers to Y.
- Let B be Z's partner in S.
- Z not rejected by any valid partner at the point when Y is rejected by A. Thus, Z prefers A to B.
- But A prefers Z to Y.
- Thus A-Z is unstable in S. ▪

since this is first rejection by a valid partner

S
Amy-Yancey
Bertha-Zeus
. . .

Stable Matching Summary

Stable matching problem. Given preference profiles of n men and n women, find a stable matching.

no man and woman prefer to be with each other than assigned partner

Gale-Shapley algorithm. Finds a stable matching in $O(n^2)$ time.

Man-optimality. In version of GS where men propose, each man receives best valid partner.

w is a valid partner of m if there exist some stable matching where m and w are paired

Q. Does man-optimality come at the expense of the women?

Woman Pessimality

Woman-pessimal assignment. Each woman receives worst valid partner.

Claim. GS finds woman-pessimal stable matching S*.

Pf.

- Suppose A-Z matched in S*, but Z is not worst valid partner for A.
- There exists stable matching S in which A is paired with a man, say Y, whom she likes less than Z.
- Let B be Z's partner in S.
- Z prefers A to B. ← man-optimality
- Thus, A-Z is an unstable in S. ▪

S
Amy-Yancey
Bertha-Zeus
. . .

Other well-known extensions

Stable roommate problem
- Single gender

College admission problem
- Multiple matchings

Hospital admission problem*
- Matching residents to hospital

Stable matching with multiple genders*
- Traditional marriage: binary
- Futuristic marriage: k-ary

Matching Residents to Hospitals

Goal. Given a set of preferences among hospitals and medical school students, design a self-reinforcing admissions process.

Unstable pair: applicant x and hospital y are unstable if:

- x prefers y to its assigned hospital.
- y prefers x to one of its admitted students.

Stable assignment. Assignment with no unstable pairs.

- Natural and desirable condition.
- Individual self-interest will prevent any applicant/hospital deal from being made.

Extensions: Matching Residents to Hospitals

Ex: Men ≈ hospitals, Women ≈ med school residents.

Variant 1. Some participants declare others as unacceptable.

resident A unwilling to work in Cleveland

Variant 2. Unequal number of men and women.

Variant 3. Limited polygamy.

hospital X wants to hire 3 residents

Variant 4. Couple

Def. Matching S unstable if there is a hospital h and resident r such that:

- h and r are acceptable to each other; and
- either r is unmatched, or r prefers h to her assigned hospital; and
- either h does not have all its places filled, or h prefers r to at least one of its assigned residents.

Application: Matching Residents to Hospitals

NRMP. (National Resident Matching Program)

- Original use just after WWII. ← predates computer usage
- Ides of March, 23,000+ residents.

Rural hospital dilemma.

- Certain hospitals (mainly in rural areas) were unpopular and declared unacceptable by many residents.
- Rural hospitals were under-subscribed in NRMP matching.
- How can we find stable matching that benefits "rural hospitals"?

Rural Hospital Theorem. Rural hospitals get exactly same residents in every stable matching!

Application: Stable Marriage with Multiple Genders

Jan. 1, 2019

Germany introduces third gender for people who identify as intersex

People who do not fit biological definition of male or female can now choose category 'diverse' on official documents

Maya Oppenheim | @mayaoppenheim | 2 days ago | 245 shares |

Like Click to follow The Independent

Jan. 3, 2019

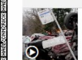
New York City birth certificates get gender-neutral option

By Evan Simko-Bednarski, CNN

Updated 5:10 PM ET, Thu January 3, 2019

New York City has changed its policy on gender designation for birth certificates.

More from CNN

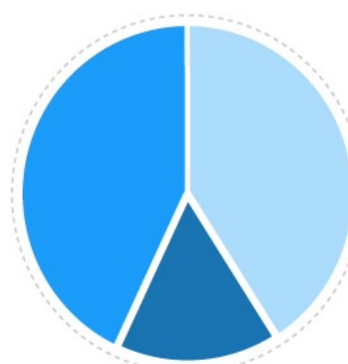


Teen charged after fatal egg-throwing prank

Apple won't be the last casualty of China's slowdown

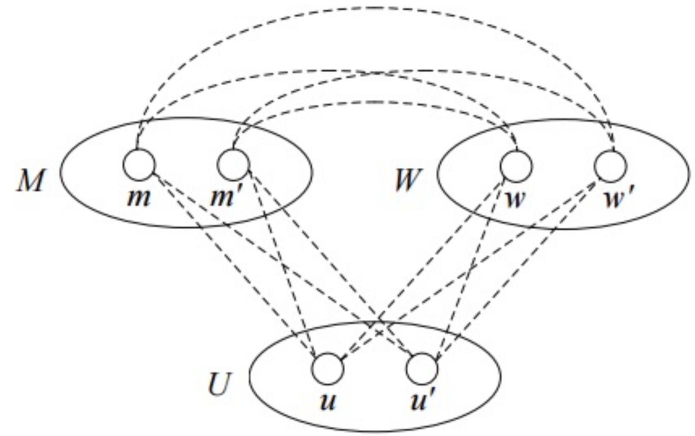
GENDER IDENTITY

People should use public restrooms according to:



Classic Marriage: stable marriage with k genders

When k=2, it is the classic marriage.



E.g., M: male, W: female, U: undecided

Theorem (Wu 2016): There exists preference lists under which there exists no stable binary matching with k (>2) genders.

Proof: Suppose u in a node (in a gender) that is ranked the lowest by all. In all other k-1 gender sets, each node x is ranked the top by exactly another node from a different gender in these k-1 genders, any marriage between u and x is unstable!

Result holds even if self-matching is allowed, e.g., nodes in U can match each other.

$$\{(m,w),(m',u),(w',u')\} \quad \{(m,w),(m',u'),(w',u)\}$$
$$\{(m,w'),(m',u),(w,u')\} \quad \{(m,w'),(m',u'),(w,u)\}$$
$$\{(m,u),(m',w),(w',u')\} \quad \{(m,u),(m',w'),(w,u')\}$$
$$\{(m,u'),(m',w),(w',u)\} \quad \{(m,u'),(m',u),(u',w)\}$$

Futuristic Marriage: stable k-ary marriage with k genders

K-ary matching: $(u_1, u_2, \ldots, u_k)$ (k: the number of genders)

Iterative Binding: Iteratively apply GS to pair wisely and bind all disjoint sets through a spanning tree.

Theorem (Wu 2016): The iterative GS constructs a stable k-ary matching.

Blocking family (for instability): if each member prefers each member of that family to its current family.

- E.g., current matching is {(m, w, u), (m', w', u')}, (m', w, u) is a blocking family if m' prefers w and u and both w and u prefers m'

J. Wu, " Stable Matching Beyond Bipartite Graphs," (APDCM) (in conjunction with IEEE IPDPS 2016)

Matching Theory in Games

Marriage with one-side domination: without money

Marriage with one-side domination: with money

The 2012 Nobel prize in economics: awarded to

Alvin E. Roth and Lloyd S. Shapley

for “the theory of stable allocations and the practice of market design”

Trading without Money: Top Trading Cycle (TTC) Algorithm

While women remain, do the following:

1. Let each remaining woman point to her favorite remaining man.
2. Reallocate as suggested by the directed cycles in the graph (including self loops) and delete the reallocated women and men. All other women keep their current men.

Four desirable properties

1. Strategy-proof
2. Pareto-optimal
3. Individual rationality
4. Unique core allocation

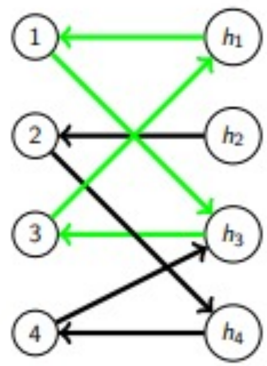
L.S. Shapley and H. Scarf, 1974, On Cores and Indivisibility. Journal of Mathematical Economics 1, 23–37.

The algorithm is described in section 6, p. 30, and attributed to David Gale

Example

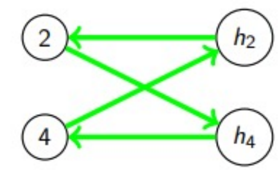
Agents' ranking from best (left) to worst (right):

$1: \quad (h_3, h_2, h_4, h_1)$
$2: \quad (h_4, h_1, h_2, h_3)$
$3: \quad (h_1, h_4, h_3, h_2)$
$4: \quad (h_3, h_2, h_1, h_4)$



Only agents 2 and 4 left with updated preferences:

$2: \quad (h_4, h_2)$
$4: \quad (h_2, h_4)$



- Cycle: $(2, h_4, 4, h_2, 2)$.
- So: 2 gets h_4 and 4 gets h_2. Done!
- Final match:

$$(1, h_3), (2, h_4), (3, h_1), (4, h_2).$$

- Cycle: $(1, h_3, 3, h_1, 1)$.
- So: 1 get h_3 and 3 gets h_1. Remove them and iterate.

Top matching with money

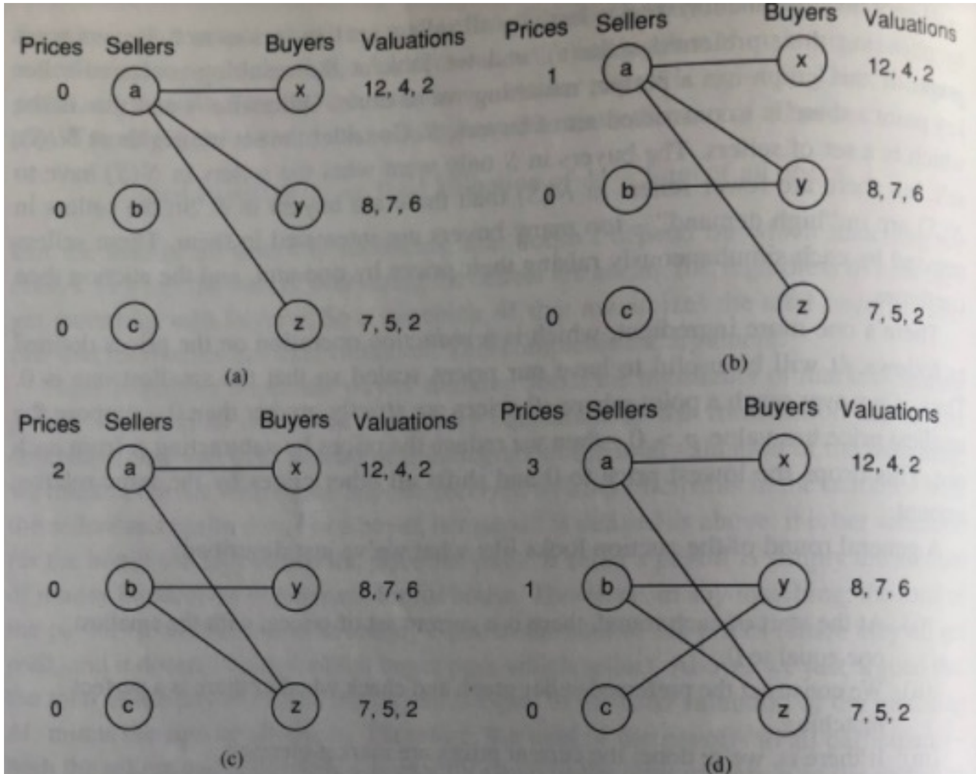


Figure 10.6. The auction procedure applied to the example from Figure 10.5. Each separate picture shows steps (i) and (ii) of successive rounds, in which the preferred-seller graph for that round is constructed. (a) In the first round, all prices start at 0. The set of all buyers forms a constricted set S, with $N(S)$ equal to the seller a. So a raises his price by one unit and the auction continues to the second round. (b) In the second round, the set of buyers consisting of x and z forms a constricted set S, with $N(S)$ again equal to seller a. Seller a again raises his price by one unit and the auction continues to the third round. (Notice that in this round, we could have alternatively identified the set of all buyers as a different constricted set S, in which case $N(S)$ would have been the set of sellers a and b. There is no problem with this – it just means that there can be multiple options for how to run the auction procedure in certain rounds, with any of these options leading to market-clearing prices when the auction comes to an end.) (c) In the third round, the set of all buyers forms a constricted set S, with $N(S)$ equal to the set of two sellers a and b. So a and b simultaneously raise their prices by one unit each, and the auction continues to the fourth round. (d) In the fourth round, when we build the preferred-seller graph, we find it contains a perfect matching. Hence, the current prices are market clearing, and the auction comes to an end.

- Perfect matching
- Market clearing price

G. Demannge, D. Gale, and M. Sotomayor, Multi-item auctions, Journal of Political Economy, 1986.