The Generalized $\mathbf{3}$-Connectivity of Some Regular Networks

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Abstract—For a vertex set S with cardinality at least two, we need a tree to connect them, where this tree is usually called an S-Steiner tree (or a tree connecting S). Two S-Steiner trees T and T' are said to be internally disjoint if $E(T) \cap E(T') = \emptyset$ and $V(T) \cap V(T') = S$. Let $\kappa_G(S)$ denote the maximum number r of internally disjoint S-Steiner trees in G. For an integer k with $2 \leq k \leq n$, the *generalized k-connectivity* of a graph G is defined as $\kappa_k(G) = min\{\kappa_G(S)|S \subseteq V(G)$ and $|S| = k\}$. It is proved NP-complete to determine $\kappa_k(G)$ for a general graph G. So far, the exact values of $\kappa_k(G)$ are known for small classes of graphs and most of them are about $k = 3$. In this paper, we introduce a family of m-regular and m-connected graph G_n which are constructed recursively and contains many important interconnection networks such as the alternating group graph AG_n, the k-ary n-cube Q^k_n, the split-star network S^2_n and the bubble-sort-star graph BS_n. We study the generalized 3-connectivity of G_n and show that $\kappa_3(G_n) = m - 1$, which attains the upper bound of $\kappa_3(G)$ given by Li *et al.* for $G = G_n$. As applications, the generalized 3-connectivity of AG_n, Q^k_n, S^2_n and BS_n etc., can be obtained directly.

Index Terms—Interconnection network; Generalized connectivity; Fault-tolerance; Regular Network.

1 INTRODUCTION

In the modern society, Big Data and Internet of Things are prevailing in computer systems and information technology. In recent years, due to the popularization of mobile devices, the prevailing of social networks and the improvement of cloud computing, enormous amount of data is produced in great speed. Internet of Things, for instance, every device is equipped with sensors. These devices are able to collect every kind of data extensively in large amount. Thus, the parallel and distributed system is an important technique for developing Big Data. Related researches about interconnection network for the most parts have applied to the parallel and distributed system. In a distributed computer system, a network structure represents the layout of the processors and the links. The topological structure of a computer network is usually represented by a graph, where vertices represent processors and edges represent links between processors. The internally disjoint S-Steiner trees of graphs do exist in information engineering design and telecommunication networks [28]. The research about internally disjoint S-Steiner trees of graphs plays a key role in effective information transportation in terms of parallel routing design for large-scale networks.

The connectivity $\kappa(G)$ is an important parameter to evaluate the reliability and fault tolerance of a graph G. As we know, $\kappa(G)$ has two equivalent definitions, one is the cut version and the other is the path version. For the cut version, it is defined as the minimum number of vertices whose deletion results in a disconnected graph. For the path version, Whitney [32] defined it from a local point of view, that is, for any subset $S = \{u, v\} \subseteq V(G)$, let $\kappa_G(S)$ denote the maximum number of internally disjoint paths between u and v in G. Then $\kappa(G) = min\{\kappa_G(S)|S \subseteq V(G)$ and $|S| = 2\}$.

The generalized k-connectivity $\kappa_k(G)$ was first mentioned by Hager [9] in 1985, it can be used to measure the reliability of a network G that connect any k vertices in G. For a vertex set S with cardinality at least two, we need a tree to connect them, where this tree is called an S-Steiner tree (or a tree connecting S). Two S-Steiner trees T and T' are said to be internally disjoint if $E(T) \cap E(T') = \emptyset$ and $V(T) \cap V(T') = S$. For $S \subseteq V(G)$ and $|S| \geq 2$, the $\kappa_G(S)$ is the maximum number of internally disjoint S-Steiner trees in G. For an integer k with $2 \leq k \leq n$, the generalized k-connectivity is defined as $\kappa_k(G) = \min\{\kappa_G(S)|S \subseteq V(G), |S| = k\}$, that is, $\kappa_k(G)$ is the minimum value of $\kappa_G(S)$ when S runs over all k-subsets of $V(G)$. Clearly, when $|S| = 2$, $\kappa_2(G)$ is just the connectivity $\kappa(G)$ of G, that is, $\kappa_2(G) = \kappa(G)$ and corresponding to the definition of $\kappa(G)$ for the path version. This is the reason why one addresses $\kappa_k(G)$ as a generalization of $\kappa(G)$.

The internally disjoint S-Steiner trees have applications in VLSI circuit design [28], that is, a Steiner tree is needed to share an electronic signal by a set of terminal nodes. In addition, the S-Steiner trees are used in computer communication networks and optical wireless communication networks, which is of prominent importance. Imagine that a given graph G represents a network. We choose arbitrary k vertices as nodes. Suppose one of the nodes in G is a broadcaster, and

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all other nodes are either users or routers (also called switches). The broadcaster wants to broadcast as many streams of movies as possible, so that the users have the maximum number of choices. Each stream of movie is broadcasted via a tree connecting all the users and the broadcaster. In essence, we need to find the maximum number of internally disjoint Steiner trees connecting all the users and the broadcaster, namely, we want to get $\kappa_G(S)$, where S is the set of the k nodes. Furthermore, if we want to know whether for any k nodes the network G has the above properties, we need to compute $\kappa_k(G)$= min$\{\kappa_G(S)\}$ in order to prescribe the reliability and the security of the network.

Determining $\kappa_k(G)$ for general graphs is a non-trivial problem. Li *et al.* [19] derived that for any fixed integer $l \geq 2$, a given graph G and a subset $S \subseteq V(G)$, deciding whether there are l internally disjoint trees connecting S, namely deciding whether $\kappa_G(S) \geq l$, is NP-complete. So far, the upper bounds and lower bounds of the generalized connectivity of graphs have been studied by the authors in Refs. [17], [21], [22]; the upper bounds and lower bounds of the generalized connectivity of Cartesian product and Lexicographic product of graphs have been studied by the authors in Refs. [13], [20]; the characterization of graphs with given generalized connectivity have been studied by the authors in Ref. [23]; the exact values of $\kappa_k(G)$ are known for small classes of graphs such as the complete graphs [6], the hypercubes [13], the star graphs and bubble-sort graphs [18], the Cayley graph generated by trees and cycles [16], the complete bipartite graphs [24], the exchanged hypercubes [40] etc.. For $k = |V(G)|$, the generalized k-connectivity of a graph G is exactly the maximum number of edge disjoint spanning trees in G. There are some results about edge disjoint spanning trees of networks [10], [12], [25], [27], [29], [31], [35]–[38]. For more results about generalized connectivity of graphs, one can refer to [14].

Overall, the exact values of $\kappa_k(G)$ are known for small classes of graphs and most of them are about $k = 3$. In this paper, we introduce a family of m-regular and m-connected graph G_n that has exactly two outside neighbors and contains many important interconnection networks such as AG_n, Q_n^k, S_n^2 and BS_n. We show that $\kappa_3(G_n) = m - 1$, which attains the upper bound of $\kappa_3(G)$ given by Li *et al.* for $G = G_n$. As applications, the generalized 3-connectivity of AG_n, Q_n^k, S_n^2 and BS_n etc., can be obtained directly.

The paper is organized as follows. In section 2, some terminologies and notations needed for the discussion are introduced. In section 3, the generalized 3-connectivity of the regular graph G_n is determined, which is the main result. In section 4, as an application of the main result, the generalized 3-connectivity of the alternating group graph AG_n, the k-ary n-cube Q_n^k, the split-star network S_n^2 and the bubble-sort-star graph BS_n etc., can be obtained directly as they are contained in G_n. In section 5, an algorithm to find the $2n-4$ internally disjoint S-Steiner trees in BS_n is presented, where $S = \{x, y, z\}$, x, y and z are any three distinct vertices of BS_n. In section 6, the limitations of the work are discussed and in section 7, the paper is concluded.

TABLE 1
Notations needed for the discussion

Notation	***Meaning***
$G = (V, E)$	A graph with vertex set V and edge set E
$\kappa(G)$	The connectivity of a graph G
$\kappa_k(G)$	The generalized k-connectivity of a graph G
$\lvert V(G)\rvert$	The order of the vertex set of a graph G
$\lvert E(G)\rvert$	The size of the edge set of a graph G
$N_G(v)$	The neighborhood of the vertex v in G
$N_G[v]$	$N_G(v) \bigcup \{v\}$, where $v \in V(G)$
$N_G(U)$	$\bigcup_{v\in U} N_G(v) - U$, where $U \subseteq V(G)$
$d_G(v)$	The degree of the vertex v in G
$\delta(G)$	The minimum degree of the graph G
$G[V']$	The subgraph induced by V' in G, where $V' \subseteq V(G)$
$[n]$	The integer set from 1 to n
Γ	A finite group
$Cay(\Gamma, S)$	The Cayley graph with vertex set Γ and edge set $\{(g, g.s) \mid g \in \Gamma, s \in S\}$, where S is a subset of Γ and the identity of the group does not belong to S.

2 Terminology and Notation

In this section, we will introduce some terminologies and notations needed for our discussion. For terminologies and notations undefined here, one can follow the reference [1]. For convenience, we use interconnection networks and graphs interchangeably.

The notations needed for our discussion are listed in Table 1 and we will introduce the terminologies needed for our discussion.

A graph is said to be *k-regular* if for any vertex v of G, $d_G(v) = k$. The (x, y)-paths P and Q in G are *internally disjoint* if they have no common internal vertices, that is $V(P) \bigcap V(Q) = \{x, y\}$. Meanwhile, two xy- paths P and Q in G are *edge disjoint* if $E(P) \cap E(Q) = \emptyset$. Let $Y \subseteq V(G)$ and $X \subset V(G) \setminus Y$, the (X, Y)-paths is a family of internally disjoint paths starting at a vertex $x \in X$, ending at a vertex $y \in Y$ and whose internal vertices belong to neither X nor Y. If $X = \{x\}$, the (X, Y)-paths is a family of internal disjoint paths whose starting vertex is x and the terminal vertices are distinct in Y, which is referred to as a *k-fan* from x to Y.

Following, we will introduce the definition of the graph G_n.

Definition 2.1. *Let n, r, a be integers and $p_i \geq 2$ be integers for $i \in [n] \setminus \{1\}$, where $r \leq a - 1$. Let G_n be an n-th regular graph, which can be constructed recursively as follows:*

(1) *The 1-th regular graph, say G_1, is a r-regular and r-connected graph with order a.*
(2) *For $n \geq 2$, the n-th regular graph, say G_n, is a regular graph that consists of p_n copies of G_{n-1}, say $G_{n-1}^1, G_{n-1}^2, \cdots, G_{n-1}^{p_n}$.*
(3) *For each $u \in V(G_{n-1}^i)$, it has two different neighbors outside G_{n-1}^i, which are called outside neighbors of u.*

In addition, the two outside neighbors of u belong to two different $(G^j_{n-1})'s$ for $j \neq i$ and $i, j \in [p_n]$.

(4) *There are same number of independent edges between G^i_{n-1} and G^j_{n-1} for $i \neq j$ and $i, j \in [p_n]$. It can be checked that there are $\frac{2ap_2p_3\cdots p_{n-1}}{p_n-1}$ cross edges between G^i_{n-1} and G^j_{n-1}.*

(5) $\frac{2ap_2p_3\cdots p_{n-1}}{p_n-1} \geq r+2(n-2)+2$, *where $r+2(n-2) \geq 4$.*

(6) *G_n is m-regular and m-connected, where $m = r+2(n-1)$.*

For convenience, let $G_n = G^1_{n-1} \bigoplus G^2_{n-1} \bigoplus \cdots \bigoplus G^{p_n}_{n-1}$. By the definition of G_n, $|G_n| = N = ap_2p_3\cdots p_n$.

3 The generalized 3-connectivity of G_n

In this section, we will study the generalized 3-connectivity of G_n. The following lemmas are useful to our main result.

In [21], Li et al. showed the following upper bound of generalized 3-connectivity of a connected graph.

Lemma 3.1. ([21]) *Let G be a connected graph and δ be its minimum degree. Then $\kappa_3(G) \leq \delta$. Further, if there are two adjacent vertices of degree δ, then $\kappa_3(G) \leq \delta - 1$.*

In [21], Li et al. showed the relationship between $\kappa(G)$ and $\kappa_3(G)$ of a connected graph.

Lemma 3.2. ([21]) *Let G be a connected graph with n vertices. If $\kappa(G) = 4k + r$, where k and r are two integers with $k \geq 0$ and $r \in \{0, 1, 2, 3\}$, then $\kappa_3(G) \geq 3k + \lceil \frac{r}{2} \rceil$. Moreover, the lower bound is sharp.*

The following lemma is a useful property of k-connected graphs.

Lemma 3.3. ([1]) *Let $G = (V, E)$ be a k-connected graph, and let X and Y be subsets of $V(G)$ of cardinality at least k. Then there exists a family of k pairwise disjoint (X, Y)-paths in G.*

In order to prove our main result, we need the following main theorems and lemmas.

Theorem 3.4. ([1]) *Let G be a k-connected graph, and let x and y be a pair of distinct vertices in G. Then there exist k internally disjoint paths $P_1, P_2, \cdots, P_k$ in G connecting x and y.*

Lemma 3.5. *(Fan Lemma [1]) Let $G = (V, E)$ be a k-connected graph, let x be a vertex of G, and let $Y \subseteq V \setminus \{x\}$ be a set of at least k vertices of G. Then there exists a k-fan in G from x to Y, that is, there exists a family of k internally disjoint (x, Y)-paths whose terminal vertices are distinct in Y.*

To prove $\kappa_3(G_n)$, the connectivity of a subgraph H of G_n is considered.

Lemma 3.6. *Let G_n and r be the same as in Definition 2.1. Let $G_n = G^1_{n-1} \bigoplus G^2_{n-1} \bigoplus \ldots \bigoplus G^{p_n}_{n-1}$ and $H = G^{i_1}_{n-1} \bigoplus G^{i_2}_{n-1} \bigoplus \ldots \bigoplus G^{i_l}_{n-1}$ be the induced subgraph of G_n on $\bigcup_{m=1}^{l} V(G^{i_m}_{n-1})$ for $2 \leq l \leq p_n - 1$. Then $\kappa(H) \geq r + 2(n-2)$, where $r + 2(n-2) \geq 4$ and $p_n \geq 3$.*

Proof: Without loss of generality, let $H = G^1_{n-1} \bigoplus G^2_{n-1} \bigoplus \ldots \bigoplus G^l_{n-1}$. To prove the result, we just need to show that there are $r + 2(n-2)$ internally disjoint paths for any two distinct vertices of H. Let $v_1, v_2 \in V(H)$ and $v_1 \neq v_2$, then the following two cases are considered.

Case 1. v_1 and v_2 belong to the same copy of G_{n-1}.

Without loss of generality, let $v_1, v_2 \in V(G^1_{n-1})$. By Definition 2.1(6), $\kappa(G^1_{n-1}) = r + 2(n-2)$. Then there are $r + 2(n-2)$ internally disjoint paths between v_1 and v_2 in G^1_{n-1}.

Case 2. v_1 and v_2 belong to two different copies of G_{n-1}.

Without loss of generality, let $v_1 \in V(G^1_{n-1})$ and $v_2 \in V(G^2_{n-1})$. Select $r + 2(n-2)$ vertices from $G^1_{n-1} \setminus \{v_1\}$, say $u_1, u_2, u_3, \cdots, u_{r+2(n-2)}$, such that the outside neighbor u'_i of u_i belongs to $G^2_{n-1} \setminus \{v_2\}$ for each $i \in [r + 2(n-2)]$. By Definition 2.1(5), this can be done. Let $S = \{u_1, u_2, u_3, \cdots, u_{r+2(n-2)}\}$ and $S' = \{u'_1, u'_2, u'_3, \cdots, u'_{r+2(n-2)}\}$. By Definition 2.1(6), $\kappa(G^1_{n-1}) = \kappa(G^2_{n-1}) = r + 2(n-2)$. By Lemma 3.5, there exists a family of $r + 2(n-2)$ internally disjoint (v_1, S)-paths $P_1, P_2, \cdots, P_{r+2(n-2)}$ such that the terminal vertex of P_i is u_i. Similarly, there exists a family of $r + 2(n-2)$ internally disjoint (v_2, S') paths $P'_1, P'_2, \cdots, P'_{r+2(n-2)}$ such that the terminal vertex of P'_i is u'_i. Let $\widehat{P}_i = P_i \bigcup u_iu'_i \bigcup P'_i$ for each $i \in [r + 2(n-2)]$, then $r+2(n-2)$ internally disjoint paths between v_1 and v_2 are obtained in H. □

In the following lemma, we will show the property of a subgraph H of G_n, which is important to prove the main result.

Lemma 3.7. *Let G_n and r be the same as in Definition 2.1 and let $H = G^{i_1}_{n-1} \bigoplus G^{i_2}_{n-1} \bigoplus G^{i_3}_{n-1} \bigoplus \cdots \bigoplus G^{i_l}_{n-1}$ be the induced subgraph of G_n on $\bigcup_{j=1}^{l} V(G^{i_j}_{n-1})$ and $x \in V(H)$, where $l \geq 2$ and $n \geq 5$. If $d_H(x) = k$ and $Y \subseteq V(H) \setminus \{x\}$ with $|Y| = k$ such that $|Y \bigcap V(G^{i_j}_{n-1})| \leq r + 2(n-2)$ for each $j \in [l]$. Then there exists a k-fan in H from x to Y.*

Proof: Without loss of generality, let $H = G^1_{n-1} \bigoplus G^2_{n-1} \bigoplus G^3_{n-1} \bigoplus \cdots \bigoplus G^l_{n-1}$. Let $x \in V(H), d_H(x) = k$ and $Y \subseteq V(H) \setminus \{x\}$ with $|Y| = k$ such that $|Y \bigcap V(G^j_{n-1})| \leq r + 2(n-2)$ for each $j \in [l]$. Clearly, $r + 2(n-2) \leq k \leq r + 2(n-1)$. To prove the result, the following three cases are considered.

Case 1. $k = r + 2(n-2)$.

By Lemma 3.6, $\kappa(H) \geq r + 2(n-2)$. By Lemma 3.5, there exists a $[r + 2(n-2)]$-fan in H from x to Y and the result is desired.

Case 2. $k = r + 2(n-1)$.

Since $d_H(x) = r+2(n-1)$, then $V(H)$ contains the two outside neighbors x' and x'' of x. By Definition 2.1(3), x' and x'' belong to different copies of G_{n-1}. Without loss of generality, let $x \in V(G^1_{n-1}), x' \in V(G^2_{n-1})$ and $x'' \in V(G^3_{n-1})$. Let $Y \bigcap V(G^j_{n-1}) = A_j$ and $|A_j| = a_j$ for $1 \leq j \leq l$. Then $a_j \leq r + 2(n-2)$ and $\sum_{j=1}^{l} a_j = r+2(n-1)$. As $|Y| = r+2(n-1)$ and $|A_1| \leq r+2(n-2)$, there are at least two vertices of Y outside G^1_{n-1}. We prove the result by considering a_j for $j = 2, 3$ and the following two subcases are considered.

Subcase 2.1. $a_2 \geq 1$ and $a_3 \geq 1$.

Let $a'_j = a_j - 1$ for $j = 2, 3$ and $a'_j = a_j$ for $j \in [l] \setminus \{2, 3\}$. Then $\sum_{j=1}^{l} a'_j = r + 2(n-2)$. Now select $l-1$ pairwise disjoint vertex sets $M_2, M_3, \cdots, M_l$ in G^1_{n-1} such that $|M_j| = a'_j$ and for any vertex v of M_j, one of the two outside neighbors of v belongs to G^j_{n-1} and $M_j \bigcap (A_1 \bigcup \{x\}) = \emptyset$ for $j \in \{2, 3, \cdots, l\}$. By Definition 2.1(5), this can be done. Let $M = A_1 \bigcup M_2 \bigcup \cdots \bigcup M_l$. As $|M| = r + 2(n-2)$ and $\kappa(G^1_{n-1}) = r + 2(n-2)$. By Lemma 3.5, there exist l fans $F_1, F_2, \cdots, F_l$ in G^1_{n-1} from x to $A_1, M_2, \cdots, M_l$, respectively, where F_1 is a family of a_1 internally disjoint (x, A_1)-paths whose terminal vertices are distinct in A_1 and F_j is a family of a'_j internally disjoint (x, M_j)-paths whose terminal vertices are distinct in M_j for $2 \leq j \leq l$. See Fig.1. Let $M'_j = \{y' | y'$ is the outside neighbor of y

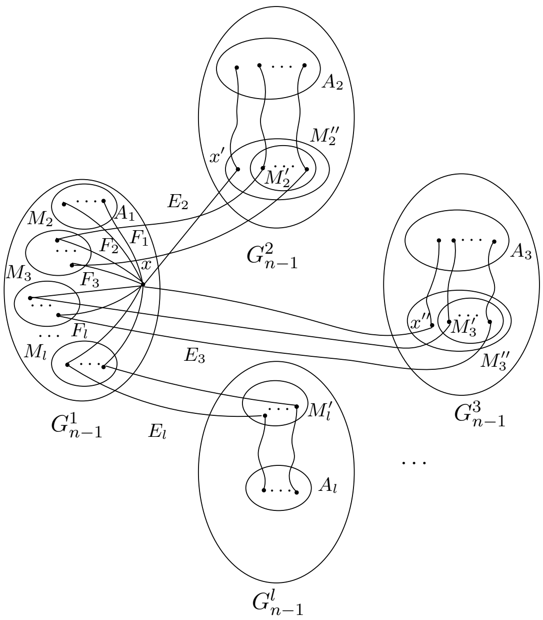

Fig. 1. Illustration of Subcase 2.1 for $A_{j0} = \emptyset$ for each $j \in \{2, 3 \cdots, l\}$ in Lemma 3.7

such that $y' \in V(G^j_{n-1})$ for each $y \in M_j\}$ and $E_j = \{yy' \in E(G_n) | y \in M_j$ and $y' \in M'_j\}$ for $2 \leq j \leq l$. Let $M''_2 = M'_2 \bigcup \{x'\}$ and $M''_3 = M'_3 \bigcup \{x''\}$, then $|M''_2| = a_2$ and $|M''_3| = a_3$. Let $M''_j \bigcap A_j = A_{j0}$ for $j = 2, 3$ and $M'_j \bigcap A_j = A_{j0}$ for $4 \leq j \leq l$. Let $M''_j \setminus A_{j0} = A_{j1}$ for $j = 2, 3$ and $M'_j \setminus A_{j0} = A_{j1}$ for $4 \leq j \leq l$, and let $A_j \setminus A_{j0} = A_{j2}$ for $2 \leq j \leq l$. Then $|A_{j1}| = |A_{j2}| = a_j - |A_{j0}|$ for $2 \leq j \leq l$. By Definition 2.1(6), $\kappa(G^j_{n-1}) = r + 2(n-2)$. As $\kappa(G^j_{n-1} \setminus A_{j0}) \geq r + 2(n-2) - |A_{j0}| \geq a_j - |A_{j0}|$. By Lemma 3.3, there exists a family of $a_j - |A_{j0}|$ pairwise disjoint (A_{j1}, A_{j2})-paths F'_j in G^j_{n-1} for $2 \leq j \leq l$.

Finally, by combining the l fans $F_1, F_2, \cdots, F_l$, the edge sets $E_2, \cdots, E_l$, the edges xx', xx'' and the paths $F'_2, \cdots, F'_l$, we can obtain a $[r + 2(n-1)]$-fan from x to Y in H.

Subcase 2.2. At least one of $a_2, a_3 = 0$.

Without loss of generality, we assume $a_2 = 0$ and the following three subcases are considered.

Subcase 2.2.1. $a_2 = 0$ and $a_3 \geq 2$.

Since $a_2 = 0$ and $a_3 \geq 2$, see Fig.2. Let $a'_j = a_j - 2$ for $j = 3$ and $a'_j = a_j$ for $j \in [l] \setminus \{3\}$. Then select $l - 2$ pairwise disjoint vertex sets $M_3, M_4, \cdots, M_l$ in G^1_{n-1} such that $|M_j| = a'_j$ and for any vertex v of M_j, one of the two outside neighbors of v belongs to G^j_{n-1} and $M_j \bigcap (A_1 \bigcup \{x\}) = \emptyset$ for each $j \in \{3, 4, \cdots, l\}$. By Definition 2.1(5), this can be done. Let $M = A_1 \bigcup M_3 \bigcup \cdots \bigcup M_l$. As $|M| = r + 2(n-2)$ and $\kappa(G^1_{n-1}) = r + 2(n-2)$ by Definition 2.1(6). By Lemma 3.5, there exist $l - 1$ fans $F_1, F_3, \cdots, F_l$ in G^1_{n-1} from x to $A_1, M_3, \cdots, M_l$, respectively. Let $M'_j = \{y' | y'$

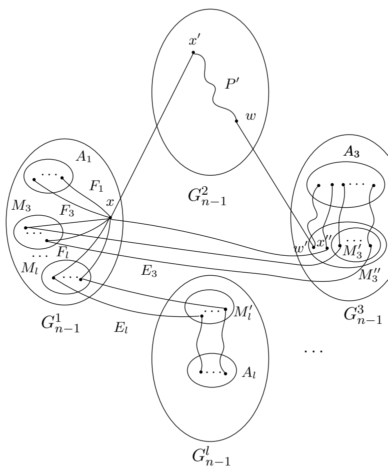

Fig. 2. Illustration of Subcase 2.2.1 for $A_{j0} = \emptyset$ for each $j \in \{3, 4 \cdots, l\}$ in Lemma 3.7

is the outside neighbor of y such that $y' \in V(G^j_{n-1})$ for each $y \in M_j\}$ and $E_j = \{yy' \in E(G_n) | y \in M_j$ and $y' \in M'_j\}$ for $3 \leq j \leq l$. Let $w \in V(G^2_{n-1})$ and one of the outside neighbors w' of w belongs to $V(G^3_{n-1})$ and $w' \notin \{x''\} \bigcup M'_3$. By Definition 2.1(5), this can be done. Then there exists a path P' between x' and w. Let $M''_3 = M'_3 \bigcup \{x'', w'\}$, then $|M''_3| = a_3$. Let $M''_j \bigcap A_j = A_{j0}$ for $j = 3$ and $M'_j \bigcap A_j = A_{j0}$ for $4 \leq j \leq l$. Let $M''_j \setminus A_{j0} = A_{j1}$ for $j = 3$ and $M'_j \setminus A_{j0} = A_{j1}$ for $4 \leq j \leq l$, and let $A_j \setminus A_{j0} = A_{j2}$ for $3 \leq j \leq l$. Then $|A_{j1}| = |A_{j2}| = a_j - |A_{j0}|$ for $3 \leq j \leq l$. By Definition 2.1 (6), $\kappa(G^j_{n-1}) = r + 2(n-2)$. We also have $\kappa(G^j_{n-1} \setminus A_{j0}) \geq r + 2(n-2) - |A_{j0}| \geq a_j - |A_{j0}|$. By Lemma 3.3, there exists a family of $a_j - |A_{j0}|$ pairwise disjoint (A_{j1}, A_{j2})-paths F'_j in AG^j_{n-1} for $3 \leq j \leq l$.

Next, by combining the $l - 1$ fans $F_1, F_3 \cdots, F_l$, the edge sets $E_3, E_4, \cdots, E_l$, the edges xx', xx'', ww', the path P' and the paths $F'_3, \cdots, F'_l$, we can obtain a $[r + 2(n-1)]$-fan from x to Y in H.

Subcase 2.2.2. $a_2 = 0$ and $a_3 = 1$.

Since $a_2 = 0$ and $a_3 = 1$, there must exist a part G^k_{n-1}

such that $a_k \geq 1$ for $k \in \{4, 5, \cdots, l\}$. Let $a'_j = a_j - 1$ for $j = 3, k$ and $a'_j = a_j$ for $j \in [l] \setminus \{3, k\}$.

Then select $l-2$ pairwise disjoint vertex sets $M_3, M_4, \cdots, M_l$ in G^1_{n-1} such that $|M_j| = a'_j$ and for any vertex v of M_j, one of the two outside neighbors of v belongs to G^j_{n-1} and $M_j \bigcap (A_1 \bigcup \{x\}) = \emptyset$ for each $j \in \{3, 4, \cdots, l\}$. Let $M = A_1 \bigcup M_3 \bigcup \cdots \bigcup M_l$. By Definition 2.1(6), $\kappa(G^1_{n-1}) = r + 2(n-3)$. As $|M| = r + 2(n-2)$, by Lemma 3.5, there exist $l-1$ fans $F_1, F_3, \cdots, F_l$ in G^1_{n-1} from x to M, where F_j is a family of a'_j internally disjoint (x, M_j)-paths whose terminal vertices are distinct in M_j for $3 \leq j \leq l$.

Let $M'_j = \{y' | y'$ is the outside neighbor of y such that $y' \in V(G^j_{n-1})$ for each $y \in M_j\}$ and $E_j = \{yy' \in E(G_n) | y \in M_j$ and $y' \in M'_j\}$ for $3 \leq j \leq l$. Let $w \in V(G^2_{n-1})$ such that one of the outside neighbors w' of w belongs to G^k_{n-1} and $w' \notin M'_k$. Then there exists a path P' from x' to w in G^2_{n-1}. Let $M''_k = M'_k \bigcup \{w'\}$ and $M''_3 = M'_3 \bigcup \{x''\}$, then $|M''_k| = a_k$ and $|M''_3| = a_3$. Then prove the result similar as Subcase 2.1, we can obtain a $[r + 2(n-1)]$-fan from x to Y in H.

Subcase 2.2.3. $a_2 = 0$ and $a_3 = 0$.

In this case, there exists a part G^k_{n-1} such that $a_k \geq 2$ for $k \in \{4, 5, \cdots, l\}$ or there exist two parts G^i_{n-1} and G^m_{n-1} such that $a_i, a_m \geq 1$ for $i, m \in \{4, 5, \cdots, l\}$.

Subcase 2.2.3.1. There exists a part G^k_{n-1} such that $a_k \geq 2$ for $k \in \{4, 5, \cdots, l\}$.

For this case, see Fig.3. Let $a'_j = a_j - 2$ for $j = k$ and $a'_j = a_j$ for $j \neq k$. Then select $l-3$ pairwise disjoint vertex sets $M_4, M_5, \cdots, M_l$ in G^1_{n-1} such that $|M_j| = a'_j$ and for any vertex v of M_j, one of the two outside neighbors of v belongs to G^j_{n-1} and $M_j \bigcap (A_1 \bigcup \{x\}) = \emptyset$ for each $j \in \{4, \cdots, l\}$. Let $M = A_1 \bigcup M_4 \bigcup \cdots \bigcup M_l$. As $|M| = r + 2(n-2)$ and $\kappa(G^1_{n-1}) = r + 2(n-2)$ by Definition 2.1(6). By Lemma 3.5, there exist $l-2$ fans $F_1, F_4, \cdots, F_l$ in G^1_{n-1} from x to M, where F_j is a family of a'_j internally disjoint (x, M_j)-paths whose terminal vertices are distinct in M_j for $4 \leq j \leq l$. Let $M'_j = \{y' | y'$ is the outside neighbor of y such that $y' \in V(G^j_{n-1})$ for each $y \in M_j\}$ and $E_j = \{yy' \in E(G_n) | y \in M_j$ and $y' \in M'_j\}$ for $4 \leq j \leq l$. Let $u \in V(G^2_{n-1})$ and one of the outside neighbors u' of u belongs to $V(G^k_{n-1})$ and $u' \notin M'_k$. Let $v \in V(G^3_{n-1})$ and one of the outside neighbors v' of v belongs to $V(G^k_{n-1})$ and $v' \notin \{u'\} \bigcup M'_k$. Then there exists a path P_1 between x' and u in G^2_{n-1} and a path P_2 between x'' and v in G^3_{n-1}. Let $M''_k = M'_k \bigcup \{u', v'\}$, then $|M''_k| = a_k$. Then prove the result similar as Subcase 2.2.1, we can obtain a $[r + 2(n-1)]$-fan from x to Y in H.

Subcase 2.2.3.2. There exist two parts G^i_{n-1} and G^m_{n-1} such that $a_i, a_m \geq 1$ for $i, m \in \{4, 5, \cdots, l\}$.

For this case, see Fig.4. Let $a'_j = a_j - 1$ for $j = i, m$ and $a'_j = a_j$ for $j \neq i, m$. Then select $l-3$ pairwise disjoint vertex sets $M_4, M_5, \cdots, M_l$ in G^1_{n-1} such that $|M_j| = a'_j$ and for any vertex v of M_j, one of the two outside neighbors of v belongs to G^j_{n-1} and $M_j \bigcap (A_1 \bigcup \{x\}) = \emptyset$ for each $j \in \{4, \cdots, l\}$. Let $M = A_1 \bigcup M_4 \bigcup \cdots \bigcup M_l$. As $|M| = r + 2(n-2)$ and $\kappa(G^1_{n-1}) = r + 2(n-2)$ by Definition 2.1(6). By Lemma 3.5, there exist $l-2$ fans $F_1, F_4, \cdots, F_l$ in G^1_{n-1} from x to M, where F_j is a family of a'_j internally disjoint (x, M_j)-paths whose terminal vertices are distinct in M_j for $4 \leq j \leq l$. Let $M'_j = \{y' | y'$ is the outside neighbor of y such that $y' \in V(G^j_{n-1})$ for each $y \in M_j\}$ and $E_j = \{yy' \in E(G_n) | y \in M_j$ and $y' \in M'_j\}$ for $4 \leq j \leq l$. Let $u \in V(G^2_{n-1})$ and one of the outside neighbors u' of u belongs to $V(G^i_{n-1})$ and $u' \notin M'_i$. Let $v \in V(G^3_{n-1})$ and one of the outside neighbors v' of v belongs to $V(G^m_{n-1})$ and $v' \notin M'_m$. Then there exists a path P_1 between x' and u in G^2_{n-1} and a path P_2 between x'' and v in G^3_{n-1}. Let $M''_i = M'_i \bigcup \{u'\}$ and $M''_m = M'_m \bigcup \{v'\}$, then $|M''_i| = a_i$ and $|M''_m| = a_m$. Then prove the result similar as Subcase 2.2.1, we can obtain a $[r + 2(n-1)]$-fan from x to Y in H.

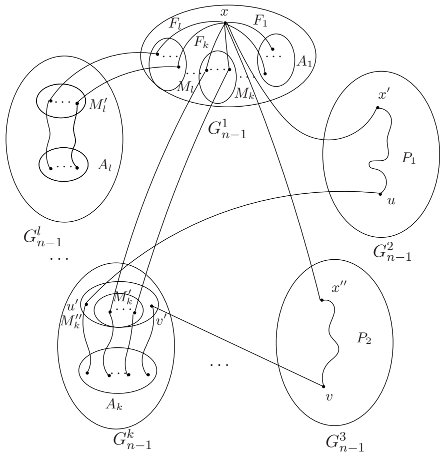

Fig. 3. Illustration of Subcase 2.2.3.1 in Lemma 3.7

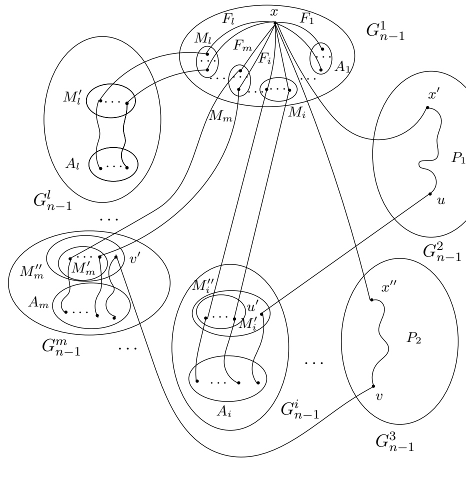

Fig. 4. Illustration of Subcase 2.2.3.2 in Lemma 3.7

Case 3. $k = r + 2n - 3$.

Since $d_H(x) = r + 2n - 3$, $V(H)$ contains one outside neighbor of x. Prove the result similar as Case 2, we can obtain a $(r+2n-3)$-fan from x to Y in H. To avoid repetition, the discussion for this case is omitted. □

In the following lemma, we will show the generalized 3-connectivity of G_n, where the three vertices in S belong to the same copy of G_{n-1}.

Lemma 3.8. *Let G_n and r be the same as in Definition 2.1, $G_n = G^1_{n-1} \bigoplus G^2_{n-1} \bigoplus \dots \bigoplus G^{p_n}_{n-1}$ and $S = \{v_1, v_2, v_3\}$, where v_1, v_2 and v_3 are any three distinct vertices of $V(G^i_{n-1})$ for $i \in [p_n]$. If there exist $r+2n-5$ internally disjoint trees connecting S in G^i_{n-1}, then there exist $r+2n-3$ internally disjoint trees connecting S in G_n.*

Proof: Without loss of generality, let $S \subseteq V(G^1_{n-1})$. Note that there exist $r+2n-5$ internally disjoint trees $T_1, T_2, \dots, T_{r+2n-5}$ connecting S in G^1_{n-1}. As v_i has two outside neighbors v'_i and v''_i for each $i \in \{1,2,3\}$ and any two distinct vertices of G^1_{n-1} have different outside neighbors by Definition 2.1(3). Hence, $M = \{v'_1, v'_2, v'_3, v''_1, v''_2, v''_3\}$ contains exactly 6 distinct vertices. In addition, each copy of G_{n-1} contains at most three vertices of them. To prove the result, the following three cases are considered.

Case 1. There exists a copy of G_{n-1} which contains three vertices of M.

Without loss of generality, let $\{v'_1, v'_2, v'_3\} \subseteq V(G^2_{n-1})$ and $\{v''_1, v''_2, v''_3\} \subseteq \bigcup_{i=3}^{p_n} V(G^i_{n-1})$. As G^2_{n-1} and $G_n[\bigcup_{i=3}^{p_n} V(G^i_{n-1})]$ as subgraphs of G_n are both connected, there is a tree, say T'_{r+2n-4}, connecting v'_1, v'_2 and v'_3 in G^2_{n-1} and a tree, say T'_{r+2n-3}, connecting v''_1, v''_2 and v''_3 in $G_n[\bigcup_{i=3}^{p_n} V(G^i_{n-1})]$, respectively. Let $T_{r+2n-4} = T'_{r+2n-4} \bigcup v_1v'_1 \bigcup v_2v'_2 \bigcup v_3v'_3$ and $T_{r+2n-3} = T'_{r+2n-3} \bigcup v_1v''_1 \bigcup v_2v''_2 \bigcup v_3v''_3$. Combine the trees T_is for $1 \le i \le r+2n-3$, then $r+2n-3$ internally disjoint trees connecting S are obtained in G_n.

Case 2. There exists a copy of G_{n-1} which contains two vertices of M and all other copies of G_{n-1} contain at most two vertices of M.

Without loss of generality, let $v'_1, v'_2 \in V(G^2_{n-1})$ and $v'_3 \in V(G^3_{n-1})$. The following two subcases are considered.

Subcase 2.1. G^3_{n-1} contains only the vertex v'_3 of $M \setminus \{v'_1, v'_2\}$.

As $G_n[\bigcup_{i=2}^{3} V(G^i_{n-1})]$ and $G_n[\bigcup_{i=4}^{p_n} V(G^i_{n-1})]$ as subgraphs of G_n are both connected, there is a tree, say T'_{r+2n-4}, connecting v'_1, v'_2 and v'_3 in $G_n[\bigcup_{i=2}^{3} V(G^i_{n-1})]$ and a tree, say T'_{r+2n-3}, connecting v''_1, v''_2 and v''_3 in $G_n[\bigcup_{i=4}^{p_n} V(G^i_{n-1})]$, respectively. Let $T_{r+2n-4} = T'_{r+2n-4} \bigcup v_1v'_1 \bigcup v_2v'_2 \bigcup v_3v'_3$ and $T_{r+2n-3} = T'_{r+2n-3} \bigcup v_1v''_1 \bigcup v_2v''_2 \bigcup v_3v''_3$. Combine the trees T_is for $1 \le i \le r+2n-3$, then $r+2n-3$ internally disjoint trees connecting S are obtained in G_n.

Subcase 2.2. G^3_{n-1} contains the vertex v'_3 and a vertex of $M \setminus \{v'_1, v'_2, v'_3\}$.

Without loss of generality, let $v'_3, v''_1 \in V(G^3_{n-1})$ and the following two subcases are considered.

Subcase 2.2.1. v''_3 and v''_2 belong to different copies of G_{n-1}.

Without loss of generality, let $v''_3 \in V(G^4_{n-1})$ and $v''_2 \in V(G^5_{n-1})$. As $G_n[V(G^2_{n-1}) \bigcup V(G^4_{n-1})]$ is connected, there is a tree, say T'_{r+2n-4}, connecting v'_1, v'_2 and v''_3 in $G_n[V(G^2_{n-1}) \bigcup V(G^4_{n-1})]$. In addition, there is a tree, say T'_{r+2n-3}, connecting v''_1, v''_2 and v'_3 in $G_n[\bigcup_{i\in[p_n]\setminus\{1,2,4\}} V(G^i_{n-1})]$ as it is connected. Let $T_{r+2n-4} = T'_{r+2n-4} \bigcup v_1v'_1 \bigcup v_2v'_2 \bigcup v_3v''_3$ and $T_{r+2n-3} = T'_{r+2n-3} \bigcup v_1v''_1 \bigcup v_2v''_2 \bigcup v_3v'_3$. Combine the trees T_is for $1 \le i \le r+2n-3$, then $r+2n-3$ internally disjoint trees connecting S are obtained in G_n.

Subcase 2.2.2. v''_3 and v''_2 belong to the same copy of G_{n-1}.

Without loss of generality, let $v''_3, v''_2 \in V(G^4_{n-1})$. As v_3 is one of the outside neighbors of v'_3 and it has exactly two outside neighbors. Then let the other outside neighbor of v'_3 be u. If $u \notin V(G^4_{n-1})$, then $G_n[\bigcup_{i\in[p_n]\setminus\{1,3,4\}} V(G^i_{n-1})]$ contains a tree T'_{r+2n-4} connecting v'_1, v'_2 and u. Let $T_{r+2n-4} = T'_{r+2n-4} \bigcup v_1v'_1 \bigcup v_2v'_2 \bigcup v_3v'_3 \bigcup v'_3u$, then it is a tree connecting S in G_n. By Lemma 3.6, $\kappa(G^3_{n-1} \bigoplus G^4_{n-1}) \ge r+2(n-2) \ge 4$. Hence, $G_n[(V(G^3_{n-1}) \bigcup V(G^4_{n-1}) \setminus \{v'_3\}]$ is connected and it contains a tree T'_{r+2n-3} connecting v''_1, v''_2 and v''_3. Let $T_{r+2n-3} = T'_{r+2n-3} \bigcup v_1v''_1 \bigcup v_2v''_2 \bigcup v_3v''_3$, then it is a tree connecting S and the result holds. Otherwise, $u \in V(G^4_{n-1})$. Let x be an in-neighbor of v'_3 in G^3_{n-1} such that one of the outside neighbors of x, say z, does not belong to G^4_{n-1}. This can be done as $r+2(n-2) \ge 4$. Hence, $G_n[\bigcup_{i\in[p_n]\setminus\{1,3,4\}} V(G^i_{n-1})]$ contains a tree, say T'_{r+2n-4}, that connects v'_1, v'_2 and z. Let $T_{r+2n-4} = T'_{r+2n-4} \bigcup v_1v'_1 \bigcup v_2v'_2 \bigcup zx \bigcup xv'_3 \bigcup v_3v'_3$, then it is a tree connecting S in G_n. By Lemma 3.6, $G_n[(V(G^3_{n-1}) \bigcup V(G^4_{n-1}) \setminus \{v'_3, x\}]$ is connected. Then there is a tree, say T'_{r+2n-3}, connecting v''_1, v''_2 and v''_3. Let $T_{r+2n-3} = T'_{r+2n-3} \bigcup v_1v''_1 \bigcup v_2v''_2 \bigcup v_3v''_3$, then it is a tree connecting S in G_n. Combine the T_is for $1 \le i \le r+2n-3$, then $r+2n-3$ internally disjoint trees connecting S in G_n are obtained.

Case 3. Each copy contains at most one vertex of M.

Without loss of generality, suppose that $G^2_{n-1}, G^3_{n-1}, G^4_{n-1}$ contains v'_1, v'_2, v'_3, respectively and $G^5_{n-1}, G^6_{n-1}, G^7_{n-1}$ contains v''_1, v''_2, v''_3, respectively. As $G_n[\bigcup_{i=2}^{4} V(G^i_{n-1})]$ and $G_n[\bigcup_{i=5}^{7} V(G^i_{n-1})]$ as induced subgraphs of G_n are both connected, there is a tree, say T'_{r+2n-4}, connecting v'_1, v'_2 and v'_3 in $G_n[\bigcup_{i=2}^{4} V(G^i_{n-1})]$ and a tree, say T'_{r+2n-3}, connecting v''_1, v''_2 and v''_3 in $G_n[\bigcup_{i=5}^{7} V(G^i_{n-1})]$, respectively. Let $T_{r+2n-4} = T'_{r+2n-4} \bigcup v_1v'_1 \bigcup v_2v'_2 \bigcup v_3v'_3$ and $T_{r+2n-3} = T'_{r+2n-3} \bigcup v_1v''_1 \bigcup v_2v''_2 \bigcup v_3v''_3$. Combine the T_is for $1 \le i \le r+2n-3$, then $r+2n-3$ internally disjoint trees connecting S in G_n are obtained. □

In the following lemma, we will show the property of a subgraph H of G_n, where there are two vertices with the same degree in H and the two vertices belong to different copies of G_{n-1}.

Lemma 3.9. *Let G_n and r be the same as in Definition 2.1 and $H = G^i_{n-1} \bigoplus G^j_{n-1}$ for $i \ne j$ and $i, j \in [p_n]$. If $x \in$*

$V(G^i_{n-1}), y \in V(G^j_{n-1})$ *and* $d_H(x) = d_H(y) = r + 2n - 3$, *then there exist* $r + 2n - 3$ *internally disjoint paths between* x *and* y *in* H.

Proof: Without loss of generality, let $H = G^1_{n-1} \bigoplus G^2_{n-1}$, $x \in V(G^1_{n-1}), y \in V(G^2_{n-1})$ and $d_H(x) = d_H(y) = r+2n-3$. To prove the main result, the following two cases are considered.

Case 1. x and y are not adjacent.

Let $Y = N_H(y) = \{y_1, y_2, \cdots, y_{r+2n-3}\}$, then $x \notin Y$. Otherwise, x and y are adjacent. Clearly, $|Y \bigcap V(G^l_{n-1})| \leq r + 2(n-2)$ for $l = 1, 2$ and $|Y| = r+2n-3$. By Lemma 3.5, there exist $r+2n-3$ internally disjoint paths $P_1, P_2, \cdots, P_{r+2n-3}$ in H from x to Y whose terminal vertices are distinct in Y. If none of the paths P_is for $1 \leq i \leq r+2n-3$ contains y as an internal vertex, then combine the edges from y to Y and the paths P_is for $1 \leq i \leq r+2n-3$, $r+2n-3$ internally disjoint paths between x and y in H can be obtained. If not, there exists only one path which contains y as an internal vertex as P_is for $1 \leq i \leq r+2n-3$ are internally disjoint. Assume that P_1 contains y as an internal vertex and the terminal vertex of P_1 is y_1. Then P_1 contains a subpath $\widetilde{P}_1$ from x to y. Combine the edges from y to $Y \setminus \{y_1\}, \widetilde{P}_1$ and the paths P_is for $2 \leq i \leq r+2n-3$, $r+2n-3$ internally disjoint (x, y)-paths in H can be obtained.

Case 2. x and y are adjacent.

Choose $r+2(n-2)$ vertices $x_1, x_2, \cdots, x_{r+2(n-2)}$ from $G^1_{n-1} \setminus \{x\}$ such that one of the outside neighbors of x_i belongs to $G^2_{n-1} \setminus \{y\}$ for each $i \in [r+2(n-2)]$. Let $X = \{x_1, x_2, \cdots, x_{r+2(n-2)}\}$ and $X' = \{x'_i | x'_i$ is the outside neighbor of x_i and $x'_i \in V(G^2_{n-1})\}$. By Definition 2.1(5), this can be done. By Definition 2.1(6), $\kappa(G^1_{n-1}) = \kappa(G^2_{n-1}) = r + 2(n-2)$. By Lemma 3.5, there exist $r+2(n-2)$ internally disjoint paths $P_1, P_2, \cdots, P_{r+2(n-2)}$ from x to X such that the terminal vertex of P_i is x_i in G^1_{n-1} and $r+2(n-2)$ internally disjoint paths $P'_1, P'_2, \cdots, P'_{r+2(n-2)}$ from y to X' such that the terminal vertex of P'_i is x'_i in G^2_{n-1} for each $i \in \{1, 2, \cdots, r+2(n-2)\}$. Let $\widetilde{P}_{r+2n-3} = xy$ and $\widetilde{P}_i = xP_ix_ix'_iP'_iy$ for $1 \leq i \leq r+2(n-2)$. Then $r+2n-3$ internally disjoint paths $\widetilde{P}_i$s for $1 \leq i \leq r+2n-3$ between x and y in H are obtained. □

Following, we will show the main result.

Theorem 3.10. *Let* G_n *and* r *be the same as in Definition 2.1 and let* $G_n = G^1_{n-1} \bigoplus G^2_{n-1} \bigoplus \ldots \bigoplus G^{p_n}_{n-1}$. *If any two vertices in different copies of* G_{n-1} *have at most one common outside neighbor, then* $\kappa_3(G_n) = r + 2n - 3$, *where* $\kappa_3(G_1) = r - 1$.

Proof: By Definition 2.1, G_n is $[r+2(n-1)]$-regular. By Lemma 3.1, $\kappa_3(G_n) \leq \delta - 1 = r+2n-3$. To prove the result, we just need to show that $\kappa_3(G_n) \geq r+2n-3$. We prove the result by induction on n.

Note that $\kappa_3(G_1) = r - 1$. Thus, the result holds for $n = 1$. Next, assume that $n \geq 2$. Let $G_n = G^1_{n-1} \bigoplus G^2_{n-1} \bigoplus \ldots \bigoplus G^{p_n}_{n-1}$ and v_1, v_2, v_3 be any three distinct vertices of G_n. For convenience, let $S = \{v_1, v_2, v_3\}$ and the following three cases are considered.

Case 1. v_1, v_2 and v_3 belong to the same copy of G_{n-1}.

Without loss of generality, let $S \subseteq V(G^1_{n-1})$. By the inductive hypothesis, there are $r+2n-5$ internally disjoint trees connecting S in G^1_{n-1}. By Lemma 3.8, there are $r+2n-3$ internally disjoint trees connecting S in G_n and the result is desired.

Case 2. v_1, v_2 and v_3 belong to two different copies of G_{n-1}.

Without loss of generality, let $v_1, v_2 \in V(G^1_{n-1})$ and $v_3 \in V(G^2_{n-1})$. By Definition 2.1(6), $\kappa(G^1_{n-1}) = r+2(n-2)$. Then there exist $r+2(n-2)$ internally disjoint paths $P_1, P_2, \ldots, P_{r+2(n-2)}$ between v_1 and v_2 in G^1_{n-1}. Let $H = G^2_{n-1} \bigoplus G^3_{n-1} \bigoplus \cdots \bigoplus G^{p_n}_{n-1}$. Then at most one outside neighbor of v_3 belongs to $V(G^1_{n-1})$ and the following two subcases are considered.

Subcase 2.1. Neither of the two outside neighbors of v_3 belong to G^1_{n-1}, that is, $d_H(v_3) = r + 2(n-1)$.

Choose $r+2(n-2)$ distinct vertices $x_1, x_2, \cdots, x_{r+2(n-2)}$ from $P_1, P_2, \ldots, P_{r+2(n-2)}$ such that $x_i \in V(P_i)$ for $1 \leq i \leq r+2(n-2)$, see Fig.5. At most one of the paths has length 1. If so, say P_1 and let $x_1 = v_1$. Let $Y = \{x_1, x_2, \cdots, x_{r+2(n-2)}\} \bigcup \{v_1, v_2\}$. If $x_1 \neq v_1$, let $Y' = \{x' | x'$ is an outside neighbor of x and $x \in Y\}$. If $x_1 = v_1$, let $Y' = \{x' | x'$ is an outside neighbor of x and $x \in Y\} \bigcup \{v''_1\}$, where v'_1 and v''_1 are two outside neighbors of v_1. Clearly, $|Y| \geq r+2n-3$ and $|Y'| = r+2(n-1)$. We can make sure that $|Y' \bigcap G^j_{n-1}| \leq r+2(n-2)$ for each $j \in \{2, 3, \cdots, p_n\}$. If not, we can replace with the other outside neighbor of x for some $x \in Y$. As $d_H(v_3) = r+2(n-1)$. By Lemma 3.5, there exist $r+2(n-1)$ internally disjoint (v_3, Y')-paths $Q_1, Q_2, \cdots, Q_{r+2(n-1)}$ in H such that the terminal vertex of Q_i is x'_i for each $i \in [r+2(n-2)]$, the terminal vertex of Q_{r+2n-3} is v'_1 or v''_1 and the terminal vertex of Q_{r+2n-2} is v'_2. Let $T_i = P_i \bigcup Q_i \bigcup x_ix'_i$ for $1 \leq i \leq r+2(n-2)$, $T_{r+2n-3} = Q_{r+2n-3} \bigcup Q_{r+2n-2} \bigcup v_2v'_2 \bigcup v_1v'_1$ or $T_{r+2n-3} = Q_{r+2n-3} \bigcup Q_{r+2n-2} \bigcup v_2v'_2 \bigcup v_1v''_1$, then $r+2n-3$ internally disjoint trees connecting S in G_n are obtained.

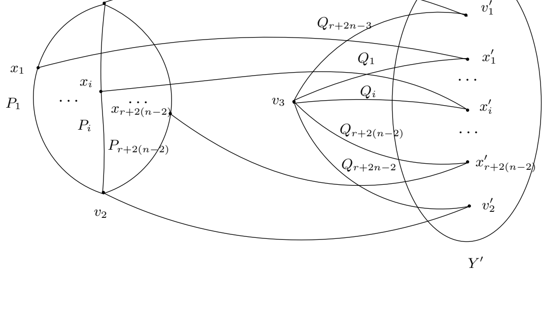

Fig. 5. Illustration of Subcase 2.1 in Theorem 3.10

Subcase 2.2. One of the outside neighbors of v_3 belongs to G^1_{n-1}, that is $d_H(v_3) = r + 2n - 3$.

Without loss of generality, let v_3' be one of the outside neighbors of v_3 and belong to G^1_{n-1}. In addition, let $V(P) = \bigcup_{i=1}^{r+2(n-2)} V(P_i)$.

If $v_3' \notin V(P)$, as G^1_{n-1} is connected, there is a (v_3', v_1)-path $\widetilde{P}$ in G^1_{n-1}. Let t be the first vertex of $\widetilde{P}$ which is in $V(P)$ and assume that $t \in V(P_{r+2(n-2)})$. Clearly, $P_{r+2(n-2)} \bigcup \widetilde{P}[v_3', t] \bigcup v_3v_3'$ is a tree connecting S, denoted by T_{r+2n-3}. If $v_3' \in V(P)$, without loss of generality, let $v_3' \in V(P_{r+2(n-2)})$. Let $T_{r+2n-3} = P_{r+2(n-2)} \bigcup v_3v_3'$, then it is a tree connecting S.

Next, choose $r + 2n - 5$ distinct vertices $x_1, x_2, \cdots, x_{r+2n-5}$ from $P_1, P_2, \ldots, P_{r+2n-5}$ such that $x_i \in V(P_i)$ for $1 \le i \le r + 2n - 5$. Denote Y and Y' similarly as in Subcase 2.1. By Lemma 3.9 and the fact that $d_H(v_3) = r + 2n - 3$, there exist $r + 2n - 3$ internally disjoint (v_3, Y')-paths $Q_1, Q_2, \cdots, Q_{r+2n-3}$ in H such that the terminal vertex of Q_i is x_i' for each $i \in [r + 2n - 5]$, the terminal vertex of Q_{r+2n-4} is v_1' or v_1'' and the terminal vertex of Q_{r+2n-3} is v_2'. Let $T_i = P_i \bigcup Q_i \bigcup x_ix_i'$ for each $i \in [r + 2n - 5]$, $T_{r+2n-4} = Q_{r+2n-4} \bigcup Q_{r+2n-3} \bigcup v_2v_2' \bigcup v_1v_1'$ or $T_{r+2n-4} = Q_{r+2n-4} \bigcup Q_{r+2n-3} \bigcup v_2v_2' \bigcup v_1v_1''$ and combining with T_{r+2n-3}, $r + 2n - 3$ internally disjoint trees connecting S in G_n are obtained.

Case 3. v_1, v_2 and v_3 belong to three different copies of G_{n-1}, respectively.

Without loss of generality, we assume that $v_1 \in V(G^1_{n-1}), v_2 \in V(G^2_{n-1})$ and $v_3 \in V(G^3_{n-1})$. Let $W = \{v_1', v_2', v_3', v_1'', v_2'', v_3''\}$, where v_i' and v_i'' are the two outside neighbors of v_i for $1 \le i \le 3$. The following three subcases are considered.

Subcase 3.1. $W \subseteq V(G^1_{n-1}) \bigcup V(G^2_{n-1}) \bigcup V(G^3_{n-1})$.

Let $H = G^1_{n-1} \bigoplus G^2_{n-1}$. Since one of the two outside neighbors of v_1 belongs to G^2_{n-1} and one of the two outside neighbors of v_2 belongs to G^1_{n-1}. Hence, $d_H(v_1) = d_H(v_2) = r + 2n - 3$. By Lemma 3.9, there exist $r + 2n - 3$ internally disjoint paths $P_1, P_2, \ldots, P_{r+2n-3}$ between v_1 and v_2 in H. Let v_3' be an outside neighbor of v_3, then $v_3' \in V(H)$. Let $V(P) = \bigcup_{i=1}^{r+2n-3} V(P_i)$, as H is connected, there is a path $\widetilde{P}$ from v_3' to v_1 in H. Let t be the first vertex of $\widetilde{P}$ which is in $V(P)$ and assume that $t \in V(P_{r+2n-3})$. Clearly, $P_{r+2n-3} \bigcup \widetilde{P}[v_3', t] \bigcup v_3v_3'$ contains a tree connecting S, denoted by T_{r+2n-3}. If $v_3' \in V(P)$, then let $v_3' \in V(P_{r+2n-3})$ and $T_{r+2n-3} = P_{r+2n-3} \bigcup v_3v_3'$, then it is a tree connecting S.

Let $x_i \in V(P_i) \bigcap N_H(v_1)$ for each $i \in [r + 2n - 4]$. If the outside neighbor of v_1 in H does not belong to x_is for $1 \le i \le r + 2n - 4$, let $X = \{x_1, x_2, \cdots, x_{r+2n-4}\}$. If the outside neighbor of v_1 in H belongs to x_is for $1 \le i \le r + 2n - 4$, say x_1, and let $X = \{v_1, x_2, \cdots, x_{r+2n-4}\}$. Then $X \subseteq V(G^1_{n-1})$ and $|X| = r + 2n - 4$. Let $H' = G^3_{n-1} \bigoplus G^4_{n-1} \bigoplus \cdots \bigoplus G^{p_n}_{n-1}$ and x_i' be one of the two outside neighbors of x_i such that $x_i' \in V(H')$ for each $i \in [r + 2n - 4]$.

If $X = \{x_1, x_2, \cdots, x_{r+2n-4}\}$, let $X' = \{x_1', x_2', \cdots, x'_{r+2n-4}\}$. By Lemma 3.1, $|X'| = r+2n-4$. As $d_{H'}(v_3) = r + 2n - 4$, by Lemma 3.5, there exist $r + 2n - 4$ internally disjoint (v_3, X')-paths $Q_1, Q_2, \cdots, Q_{r+2n-4}$ in H' such that the terminal vertex of Q_i is x_i' for each $i \in [r + 2n - 4]$. Note that at most one of Q_is for $1 \le i \le r + 2n - 4$ has length one. Let $T_i = P_i \bigcup Q_i \bigcup x_ix_i'$ for $1 \le i \le r + 2n - 4$. Combining with T_is for $1 \le i \le r + 2n - 3$, then $r + 2n - 3$ internally disjoint trees connecting S in G_n are obtained.

If $X = \{v_1, x_2, \cdots, x_{r+2n-4}\}$, let $X' = \{v_1', x_2', \cdots, x'_{r+2n-4}\}$, where $v_1' \in V(H')$. With the similar method as $X = \{x_1, x_2, \cdots, x_{r+2n-4}\}$, $r + 2n - 3$ internally disjoint trees T_is for $1 \le i \le r + 2n - 3$ connecting S in G_n can be obtained.

Subcase 3.2. $W \nsubseteq V(G^1_{n-1}) \bigcup V(G^2_{n-1}) \bigcup V(G^3_{n-1})$.

Since $W \nsubseteq V(G^1_{n-1}) \bigcup V(G^2_{n-1}) \bigcup V(G^3_{n-1})$, at least one of the outside neighbors of v_3 does not belong to $V(G^1_{n-1}) \bigcup V(G^2_{n-1})$. Let $H = G^1_{n-1} \bigoplus G^2_{n-1}$ and $H' = G^3_{n-1} \bigoplus G^4_{n-1} \bigoplus \cdots \bigoplus G^{p_n}_{n-1}$. Then select $r + 2n - 4$ vertices from $G^1_{n-1} \setminus \{v_1\}$, say $x_1, x_2, \cdots, x_{r+2n-4}$, such that one of the outside neighbors x_i' of x_i belongs to G^2_{n-1} for each $i \in [r + 2n - 4]$. Further, we request that x_i and v_2 have different outside neighbors for $1 \le i \le r+2n-4$.

Let $S = \{x_1, x_2, \cdots, x_{r+2n-4}\}$ and $S' = \{x_1', x_2', \cdots, x'_{r+2n-4}\}$. By Definition 2.1(6), $\kappa(G^1_{n-1}) = \kappa(G^2_{n-1}) = r + 2n - 4$. By Lemma 3.5, there exist $r + 2n - 4$ internally disjoint (v_1, S)-paths $P_1, P_2, \ldots, P_{r+2n-4}$ in G^1_{n-1} such that the terminal vertex of P_i is x_i and there exist $r + 2n - 4$ internally disjoint (v_2, S')-paths $P_1', P_2', \ldots, P'_{r+2n-4}$ in G^2_{n-1} such that the terminal vertex of P_i' is x_i' for $1 \le i \le r+2n-4$. Thus, we obtain $r + 2n - 4$ internally disjoint paths between v_1 and v_2 in H, where $\widetilde{P}_i = v_1P_ix_ix_i'P_i'v_2$ for each $i \in [r + 2n - 4]$.

Now, let v_i'' be one of the outside neighbors of v_i such that $v_i'' \in V(H')$ for $i = 1, 2$ and x_i'' be the other outside neighbor of x_i such that $x_i'' \in V(H')$ for $1 \le i \le r + 2n - 4$. Let $Y = \{x_1'', x_2'', x_3'', \cdots, x''_{r+2n-4}, v_1'', v_2''\}$. Then $Y \subseteq V(H')$ and $|Y| \ge r + 2n - 3$. If $v_1'' \ne v_2''$, then $|Y| = r + 2n - 2$. If $v_1'' = v_2''$, then $|Y| = r + 2n - 3$.

Subcase 3.2.1. Neither of the two outside neighbors of v_3 belong to $\bigcup_{i=1}^2 V(G^i_{n-1})$.

In this case, $d_{H'}(v_3) = r + 2n - 2$. If $|Y| = r + 2n - 2$, the proof is similar as Subcase 2.1. If $|Y| = r + 2n - 3$, the proof is similar as Subcase 2.1 except that the paths Q_{r+2n-3} and Q_{r+2n-2} become the same path.

Subcase 3.2.2. One of the two outside neighbors of v_3 belongs to $\bigcup_{i=1}^2 V(G^i_{n-1})$.

In this case, $d_{H'}(v_3) = r + 2n - 3$. If $|Y| = r + 2n - 2$, the proof is similar to Subcase 2.2. If $|Y| = r+2n-3$, the proof is also similar to Subcase 2.2 except that the paths Q_{r+2n-4} and Q_{r+2n-3} become the same path.

Hence, $r + 2n - 3$ internally disjoint trees connecting S in G_n can be obtained and the result is desired. □

4 Applications

In this section, we will present the usefulness of the main result. As an application of Theorem 3.10, the generalized 3-connectivity of AG_n, Q_n^k, S_n^2 and BS_n etc., can be obtained directly as they can be regarded as special examples of G_n.

4.1 Application to the alternating group graph $\mathbf{AG_n}$

The alternating group graph was introduced by Jwo *et al.* [11] in 1993. It is defined as follows.

Definition 4.1. *Let A_n be the alternating group of order n with $n \geq 3$ and let $S = \{(12i), (1i2) | 3 \leq i \leq n\}$. The alternating group graph, denoted by AG_n, is defined as the Cayley graph $Cay(A_n, S)$.*

By the definition of AG_n, it is a $2(n-2)$-regular graph with $n!/2$ vertices. Let A_n^i be the subset of A_n that consists of all even permutations with element i in the rightmost position and let AG_{n-1}^i be the subgraph of AG_n induced by A_n^i for $i \in [n]$. Then AG_{n-1}^i is isomorphic to AG_{n-1} for each $i \in [n]$ and we call such an AG_{n-1}^i a copy of AG_{n-1}. Thus, AG_n can be decomposed into n copies of AG_{n-1}, namely, $AG_{n-1}^1, AG_{n-1}^2, \cdots, AG_{n-1}^n$. For convenience, we denote $AG_n = AG_{n-1}^1 \bigoplus AG_{n-1}^2 \bigoplus \cdots \bigoplus AG_{n-1}^n$, where $\bigoplus$ just denotes the corresponding decomposition of AG_n. For each vertex $u \in V(AG_{n-1}^i)$, it has $2(n-3)$ neighbors in AG_{n-1}^i and two neighbors outside AG_{n-1}^i, which are called the outside neighbors of u. The graph AG_4 is depicted in Fig. 6.

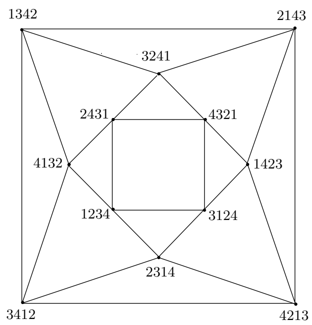

Fig. 6. The alternating group graph AG_4 of Definition 4.1

The following lemmas are about properties of AG_n.

Lemma 4.2. ([39]) *Let $AG_n = AG_{n-1}^1 \bigoplus AG_{n-1}^2 \bigoplus \ldots \bigoplus AG_{n-1}^n$ for $n \geq 3$. Then the following results hold.*

(1) *For any vertex u of AG_{n-1}^i for $i \in [n]$, it has two outside neighbors.*
(2) *For each copy AG_{n-1}^i, no two vertices in AG_{n-1}^i have a common outside neighbor. In addition, $|N(AG_{n-1}^i)| = (n-1)!$ and $|N(AG_{n-1}^i) \bigcap V(AG_{n-1}^j)| = (n-2)!$ for $i \neq j$ and $i, j \in [n]$.*

Lemma 4.3. ([11], [35]) $\kappa(AG_n) = 2(n-2)$ *for* $n \geq 3$.

Lemma 4.4. ([39]) *Let $AG_n = AG_{n-1}^1 \bigoplus AG_{n-1}^2 \bigoplus \ldots \bigoplus AG_{n-1}^n$ for $n \geq 3$. Then any two vertices in different copies of AG_{n-1} have at most one common outside neighbor.*

Corollary 4.5. $\kappa_3(AG_n) = 2n - 5$ *for* $n \geq 3$.

Proof: By Definition 2.1, AG_n can be regarded as the special regular graph G_{n-2} with $G_1 = AG_3$, $a = 3$, $r = 2$, $s = 2$, $p_{n-2} = n$ and $N = ap_2p_3 \cdots p_{n-2} = \frac{n!}{2}$. By Lemma 4.3, $\kappa(AG_3) = 2$. By Lemma 3.1, $\kappa_3(AG_3) \leq 1$. By Lemma 3.2, $\kappa_3(AG_3) \geq 1$. Thus, $\kappa_3(AG_3) = 1$. Thus, by Lemma 4.4 and Theorem 3.10, $\kappa_3(AG_n) = 2n - 5$ for $n \geq 3$. □

4.2 Application to the $\mathbf{k}$-ary $\mathbf{n}$-cube $\mathbf{Q_n^k}$

The k-ary n-cube network, denoted by Q_n^k, was introduced by S. Scott *et al.* [30] in 1994. It is defined as follows.

Definition 4.6. *The k-ary n-cube, denoted by Q_n^k, where $k \geq 2$ and $n \geq 1$ are integers, is a graph consisting of k^n vertices, each of these vertices has the form $u = u_{n-1}u_{n-2} \cdots u_0$, where $u_i \in \{0, 1, \cdots, k-1\}$ for $0 \leq i \leq n-1$. Two vertices $u = u_{n-1}u_{n-2} \cdots u_0$ and $v = v_{n-1}v_{n-2} \cdots v_0$ in Q_n^k are adjacent if and only if there exists an integer j, where $0 \leq j \leq n-1$, such that $u_j = v_j \pm 1 (mod\ k)$ and $u_i = v_i$ for every $i \in \{0, 1, \cdots, k-1\} \setminus \{j\}$. In this case, (u, v) is a j-dimensional edge.*

By the definition of Q_n^k, it is $2n$-regular for $k \geq 3$ and n-regular for $k = 2$. Clearly, Q_1^k is a cycle of length k and Q_n^2 is the hypercube.

The k-ary n-cube Q_n^k can be partitioned into k disjoint subcubes along the jth-dimension for $j \in \{0, 1, 2, \cdots, n-1\}$, namely, $Q_{n-1}^k[0], Q_{n-1}^k[1], \cdots, Q_{n-1}^k[k-1]$. Then $Q_{n-1}^k[i]$ is isomorphic to the k-ary $(n-1)$-cube for $i \in \{0, 1, 2, \cdots, k-1\}$. For convenience, we denote $Q_n^k = Q_{n-1}^k[0] \bigoplus Q_{n-1}^k[1] \bigoplus \cdots \bigoplus Q_{n-1}^k[k-1]$, where $\bigoplus$ just denotes the corresponding decomposition of Q_n^k. For each vertex $u \in V(Q_{n-1}^k[i])$, it has $2n-2$ neighbors in $Q_{n-1}^k[i]$ and two neighbors outside $Q_{n-1}^k[i]$, which are called the outside neighbors of u. The graph Q_2^4 is depicted in Fig. 7.

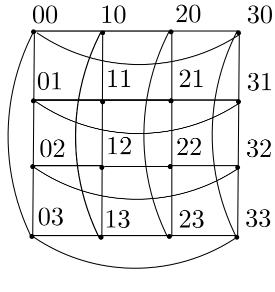

Fig. 7. The 4-ary 2-cube Q_2^4 of Definition 4.6

The following lemmas are about properties of Q_n^k.

Lemma 4.7. *Let $Q_n^k = Q_{n-1}^k[0] \bigoplus Q_{n-1}^k[1] \bigoplus \dots \bigoplus Q_{n-1}^k[k-1]$ for $k \geq 3$ and $n \geq 1$. Then the following results hold.*

(1) *For any vertex u of $Q_{n-1}^k[i]$, it has exactly two outside neighbors, where $0 \leq i \leq k-1$.*
(2) *The outside neighbors of u belong to different copies of Q_{n-1}^k. That is, no two vertices in Q_{n-1}^k have a common outside neighbor.*
(3) *$|N(Q_{n-1}^k[i])| = 2k^{n-1}$ and $|N(Q_{n-1}^k[i]) \bigcap V(Q_{n-1}^k[j])| = \frac{2k^{n-1}}{k-1}$ for $i \neq j$ and $0 \leq i, j \leq k-1$. That is, there are $\frac{2k^{n-1}}{k-1}$ independent crossed edges between two different $Q_{n-1}^k[i]$s.*

Proof: (1) Let $u = u_1u_2u_3 \cdots u_{n-1}i \in V(Q_{n-1}^k[i])$, where $0 \leq i \leq k-1$. By Definition 4.6, $u' = u_1u_2u_3 \cdots u_{n-1}(i-1)$ and $u'' = u_1u_2u_3 \cdots u_{n-1}(i+1)$ are the two outside neighbors of u.

(2) Let $u = u_1u_2u_3 \cdots u_{n-1}i \in V(Q_{n-1}^k[i])$, where $0 \leq i \leq k-1$. By (1), $u' \in V(Q_{n-1}^k[i-1])$ and $u'' \in V(Q_{n-1}^k[i+1])$. As $k \geq 3$, then $i-1 \neq i+1$. Thus, u' and u'' belong to different copies of Q_{n-1}^k.

(3) As any vertex of $Q_{n-1}^k[i]$ has two outside neighbors and $|Q_{n-1}^k[i]| = k^{n-1}$ for $0 \leq i \leq k-1$, then $|N(Q_{n-1}^k[i])| = 2k^{n-1}$ and $|N(Q_{n-1}^k[i]) \bigcap V(Q_{n-1}^k[j])| = \frac{2k^{n-1}}{k-1}$ for $i \neq j$ and $0 \leq i, j \leq k-1$. □

Lemma 4.8. ([8]) *$\kappa(Q_n^k) = 2n$ for $k \geq 3$ and $n \geq 1$.*

Lemma 4.9. *Let $Q_n^k = Q_{n-1}^k[0] \bigoplus Q_{n-1}^k[1] \bigoplus \dots \bigoplus Q_{n-1}^k[k-1]$ for $k \geq 3$ and $n \geq 1$. Then any two vertices in different copies of Q_{n-1}^k have at most one common outside neighbor.*

Proof: Let $u, v \in V(Q_n^k), u \neq v$ and they belong to different copies of Q_{n-1}^k. Without loss of generality, let $u = u_1u_2u_3 \cdots u_{n-1}0 \in V(Q_{n-1}^k[0])$ and $v = v_1v_2v_3 \cdots v_{n-1}1 \in V(Q_{n-1}^k[1])$. Then the two outside neighbors of u are $u' = u_1u_2u_3 \cdots u_{n-1}1$ and $u'' = u_1u_2u_3 \cdots u_{n-1}(k-1)$, and the two outside neighbors of v are $v' = v_1v_2v_3 \cdots v_{n-1}0$ and $v'' = v_1v_2v_3 \cdots v_{n-1}2$. If u and v have two common outside neighbors, then $\{u', u''\} = \{v', v''\}$. As $u' \neq v'$, then $u' = v''$ and $v' = u''$. However, $u' \neq v''$ clearly, which is a contradiction. Thus, u and v have at most one common outside neighbor. □

Corollary 4.10. *$\kappa_3(Q_n^k) = 2n-1$ for $k \geq 3$ and $n \geq 1$.*

Proof: By Definition 2.1, $Q_n^k (k \geq 3)$ can be regarded as the special regular graph G_n with $G_1 = Q_1^k$, $a = k$, $r = 2$, $s = 2$, $p_n = k$ and $N = ap_2p_3 \cdots p_n = k^n$. By Lemma 4.8, $\kappa(Q_1^k) = 2$. By Lemma 3.1, $\kappa_3(Q_1^k) \leq 1$. By Lemma 3.2, $\kappa_3(Q_1^k) \geq 1$. Thus, $\kappa_3(Q_1^k) = 1$. By Lemma 4.9 and Theorem 3.10, $\kappa_3(Q_n^k) = 2n-1$ for $k \geq 3$ and $n \geq 1$. □

4.3 Application to the split-star network $\mathbf{S_n^2}$

The split-star network, denoted by S_n^2, was proposed by E. Cheng *et al.* [5] as an attractive variation of the star graph in 1998. It is defined as follows, where the description has a slight modification.

Definition 4.11. *Let $Sym(n)$ be symmetric group on $[n]$ and let $S = \{(12)\} \bigcup \{(12i), (1i2) | 3 \leq i \leq n\}$. The split-star network, denoted by S_n^2, is defined as the Cayley graph $Cay(Sym(n), S)$.*

By the definition of S_n^2, it is a $(2n-3)$-regular graph with $n!$ vertices. Let $V_n^{n:i}$ be the set of vertices in S_n^2 with the n-th position being i, that is, $V_n^{n:i} = \{u | u = u_1u_2 \cdots u_{n-1}i\}$. The set $\{V_n^{n:i} | 1 \leq i \leq n\}$ forms a partition of $V(S_n^2)$. Let $S_{n-1}^2[i]$ be the subgraph of S_n^2 induced by $V_n^{n:i}$. Then $S_{n-1}^2[i]$ is isomorphic to S_{n-1}^2 and we call such an $S_{n-1}^2[i]$ a copy of S_{n-1}^2. Thus, S_n^2 can be decomposed into n copies of S_{n-1}^2, namely, $S_{n-1}^2[1], S_{n-1}^2[2], \cdots, S_{n-1}^2[n]$. For convenience, we denote $S_n^2 = S_{n-1}^2[1] \bigoplus S_{n-1}^2[2] \bigoplus \dots \bigoplus S_{n-1}^2[n]$, where $\bigoplus$ just denotes the corresponding decomposition of S_n^2. For each vertex $u \in V(S_{n-1}^2[i])$, it has $2n-5$ neighbors in $S_{n-1}^2[i]$ and two neighbors outside $S_{n-1}^2[i]$, which are called outside neighbors of u. The graph S_4^2 is depicted in Fig. 8.

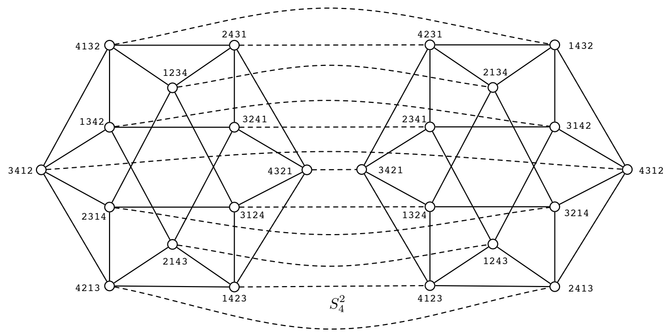

Fig. 8. The split-star network S_4^2 of Definition 4.11

The following lemmas are about properties of S_n^2.

Lemma 4.12. ([4]) *Let $S_n^2 = S_{n-1}^2[1] \bigoplus S_{n-1}^2[2] \bigoplus \dots \bigoplus S_{n-1}^2[n]$ for $n \geq 3$. Then the following results hold.*

(1) *For any vertex u of $S_{n-1}^2[i]$, it has exactly two outside neighbors, where $i \in [n]$.*
(2) *The outside neighbors of u belong to different copies of S_{n-1}^2. That is, no two vertices in $S_{n-1}^2[i]$ have a common outside neighbor for $i \in [n]$.*
(3) *$|N(S_{n-1}^2[i])| = 2(n-1)!$ and $|N(S_{n-1}^2[i]) \bigcap V(S_{n-1}^2[j])| = 2(n-2)!$ for $i \neq j$ and $i, j \in [n]$. That is, there are $2(n-2)!$ independent crossed edges between two different BS_{n-1}^is.*

Lemma 4.13. ([4]) *$\kappa(S_n^2) = 2n-3$ for $n \geq 3$.*

Lemma 4.14. *Let $S_n^2 = S_{n-1}^2[1] \bigoplus S_{n-1}^2[2] \bigoplus \dots \bigoplus S_{n-1}^2[n]$ for $n \geq 3$. Then any two vertices in different copies of S_{n-1}^2 have at most one common outside neighbor.*

Proof: Let $u, v \in V(S_n^2), u \neq v$ and they belong to different copies of S_{n-1}^2. Without loss of generality, let $u = u_1u_2u_3 \cdots u_{n-1}1 \in V(S_{n-1}^2[1])$ and $v = v_1v_2v_3 \cdots v_{n-1}2 \in V(S_{n-1}^2[2])$. Then the two outside neighbors of u are $u' = u(12n) = u_2 1 u_3 \cdots u_{n-1}u_1$ and $u'' = u(1n2) = 1u_1u_3 \cdots u_{n-1}u_2$, and the two outside neighbors of v are $v' = v(12n) = v_2 2 v_3 \cdots v_{n-1}v_1$ and $v'' = v(1n2) = 2v_1v_3 \cdots v_{n-1}v_2$. If u and v have two common outside neighbors, then $\{u', u''\} = \{v', v''\}$. As $u' \neq v'$, then $u' = v''$ and $v' = u''$. By $u' = v''$, we have that $u_2 = 2$ and $v_1 = 1$. By $u'' = v'$, we have that $v_2 = 1$ and $u_1 = 2$. That is, $u_1 = u_2 = 2$, which is a contradiction. Thus, u and v have at most one common outside neighbor. □

Corollary 4.15. *$\kappa_3(S_n^2) = 2n - 4$ for $n \geq 3$.*

Proof: By Definition 2.1, S_n^2 can be regarded as the special regular graph G_{n-2} with $G_1 = S_3^2$, $a = 6$, $r = 3$, $s = 2$, $p_{n-2} = n$ and $N = ap_2p_3 \cdots p_{n-2} = n!$. By Lemma 4.13, $\kappa(S_3^2) = 3$. By Lemma 3.1, $\kappa_3(S_3^2) \leq 2$. By Lemma 3.2, $\kappa_3(S_3^2) \geq 2$. Thus, $\kappa_3(S_3^2) = 2$. Thus, by Lemma 4.14 and Theorem 3.10, $\kappa_3(S_n^2) = 2n-4$ for $n \geq 3$. □

4.4 Application to the bubble-sort-star network $\mathbf{BS_n}$

The bubble-sort star graph, denoted by BS_n, was introduced by Z. Chou *et al.* [7] in 1996. It is defined as follows.

Definition 4.16. *Let $Sym(n)$ be symmetric group on $[n]$ and let $S = \{(1i)|2 \leq i \leq n\} \bigcup \{(i, i+1)|2 \leq i \leq n-1\}$. The n-dimensional bubble-sort star graph, denoted by BS_n, is defined as the Cayley graph $Cay(Sym(n), S)$.*

By the definition of BS_n, it is a $(2n-3)$-regular graph with $n!$ vertices. For an integer $i \in [n]$, let BS_{n-1}^i be the graph induced by the vertex set $\{p_1p_2 \cdots p_{n-1}i\}$, where $p_1p_2 \cdots p_{n-1}$ ranges over all the permutations of $\{1, 2, \cdots, i-1, i+1, \cdots, n\}$. Then BS_{n-1}^i is isomorphic to BS_{n-1} for each $i \in [n]$ and we call such an BS_{n-1}^i a copy of BS_{n-1}. Thus, BS_n can be decomposed into n copies of BS_{n-1}, namely, $BS_{n-1}^1, BS_{n-1}^2, \cdots, BS_{n-1}^n$. For convenience, let $BS_n = BS_{n-1}^1 \bigoplus BS_{n-1}^2 \bigoplus \cdots BS_{n-1}^n$. For each vertex $u \in V(BS_{n-1}^i)$, it has $2n-5$ neighbors in BS_{n-1}^i and two neighbors outside BS_{n-1}^i, which are called the outside neighbors of u. The graph BS_2 and BS_3 are depicted in Fig. 9, respectively.

The following lemmas are about properties of BS_n.

Lemma 4.17. ([3], [33]) *Let $BS_n = BS_{n-1}^1 \bigoplus BS_{n-1}^2 \bigoplus \ldots \bigoplus BS_{n-1}^n$, where $n \geq 4$. Then the following results hold.*

(1) *For any vertex u of BS_{n-1}^i, it has exactly two outside neighbors, where $i \in [n]$.*
(2) *For any vertex u of BS_n, the outside neighbors of u belong to different copies of BS_{n-1}. That is, no two vertices in BS_{n-1}^i have a common outside neighbor for $i \in [n]$.*

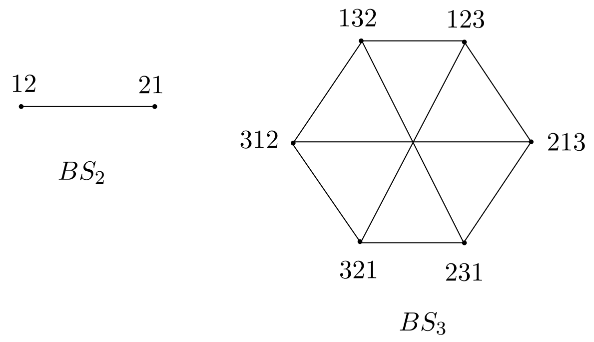

Fig. 9. The bubble-sort star graphs BS_2 and BS_3 of Definition 4.16

(3) *There are $2(n-2)!$ independent crossed edges between two different BS_{n-1}^is.*

Lemma 4.18. ([3]) *$\kappa(BS_n) = 2n - 3$ for $n \geq 3$.*

Lemma 4.19. *Let $BS_n = BS_{n-1}^1 \bigoplus BS_{n-1}^2 \bigoplus \ldots \bigoplus BS_{n-1}^n$. Then any two vertices in different copies of BS_{n-1} have at most one common outside neighbor.*

Proof: Let $u, v \in V(BS_n), u \neq v$ and they belong to different copies of BS_{n-1}. Without loss of generality, let $u = u_1u_2u_3 \cdots u_{n-1}1 \in V(BS_{n-1}^1)$ and $v = v_1v_2v_3 \cdots v_{n-1}2 \in V(BS_{n-1}^2)$. Then the two outside neighbors of u are $u' = u(1n) = 1u_2u_3 \cdots u_{n-1}u_1$ and $u'' = u(n-1, n) = u_1u_2 \cdots u_{n-2}1u_{n-1}$, and the two outside neighbors of v are $v' = v(1n) = 2v_2v_3 \cdots v_{n-1}v_1$ and $v'' = v(n-1, n) = v_1v_2 \cdots v_{n-2}2v_{n-1}$. If u and v have two common outside neighbors, then $\{u', u''\} = \{v', v''\}$. As $u' \neq v'$, then $u' = v''$ and $v' = u''$. By $u' = v''$, we have that $v_1 = 1$ and $u_{n-1} = 2$. By $u'' = v'$, we have that $v_{n-1} = 1$ and $u_1 = 2$. That is, $u_1 = u_{n-1} = 2$, which is a contradiction. Thus, u and v have at most one common outside neighbor. □

Corollary 4.20. *$\kappa_3(BS_n) = 2n - 4$ for $n \geq 3$.*

Proof: By Definition 2.1, BS_n can be regarded as the special regular graph G_{n-2} with $G_1 = BS_3$, $a = 6$, $r = 3$, $s = 2$, $p_{n-2} = n$ and $N = ap_2p_3 \cdots p_{n-2} = n!$. By Lemma 4.18, $\kappa(BS_3) = 3$. By Lemma 3.1, $\kappa_3(BS_3) \leq 2$. By Lemma 3.2, $\kappa_3(BS_3) \geq 2$. Thus, $\kappa_3(BS_3) = 2$. Thus, by Lemma 4.19 and Theorem 3.10, $\kappa_3(BS_n) = 2n - 4$ for $n \geq 3$. □

5 An algorithm for $\mathbf{BS_n}$

In this section, we will present an algorithm to find the $2n - 4$ internally disjoint S-Steiner trees in BS_n for $S = \{x, y, z\} \subseteq V(BS_n)$. To present the algorithm, the following lemmas are useful.

Lemma 5.1. ([1]) *There exists a Kruskal algorithm for finding a spanning tree in any connected graph G with n vertices, denoted by $INT(G, S)$, where $S \subseteq V(G)$.*

Lemma 5.2. ([26]) *There exists an algorithm for finding the maximum number of internally disjoint paths between two vertex set of a connected graph G.*

In order to express the algorithm compactly, we denote some notations needed for the algorithm. For $v \in V(BS_n)$, let v' and v'' be the two outside neighbors of v. In addition, let $(v)_n$ be the n-th bit number of v in BS_n.

Algorithm 1 IDT(BS_n, n, r, x, y, z)

Input: Any three distinct vertices x, y and z of BS_n and $r = 2n-4$, where $S = \{x, y, z\}$.
Output: $2n-4$ internally disjoint S-Steiner trees $T_1, T_2, \cdots, T_{2n-4}$ such that $E(T_i) \cap E(T_j) = \emptyset$ and $V(T_i) \cap V(T_j) = S$.

$\alpha \leftarrow (x)_n$, $\beta \leftarrow (y)_n$, $\gamma \leftarrow (z)_n$, $n' \leftarrow n-1$, $r' \leftarrow r-2$, $[n] \leftarrow \{1, \dots, n\}$; $\tau \leftarrow (x')_n$, $\tau' \leftarrow (x'')_n$, $\theta \leftarrow (y')_n$, $\theta' \leftarrow (y'')_n$, $\eta \leftarrow (z')_n$, $\eta' \leftarrow (z'')_n$, $M = \{x', x'', y', y'', z', z''\}$, $M \cap V(BS^i_{n-1}) = M_i$, $|M_i| = \sigma(i)$, $\sigma(\tau)$=max$\{\sigma(i)|i \in [n]\}$, $G_I = BS_n[\cup_{i \in I} V(BS^i_{n-1})]$ and $G'_I = BS_n[\cup_{i\in I} V(BS^i_{n-1}) \cup S]$, where $I \subseteq [n]$.
if $\alpha = \beta = \gamma$ **then**
  $\{T_i | 1 \le i \le r'\} \leftarrow IDT(BS^\alpha_{n-1}, n', r', x, y, z)$;
  if $\sigma(\tau) = 3$ **then**
    $M_\tau \leftarrow \{x', y', z'\}$,
    $\cup_{i \in [n] \setminus \{\alpha, \tau\}} M_i \leftarrow \{x'', y'', z''\}$,
    $T_{2n-5} \leftarrow INT(G'_{\{\tau\}}, S)$,
    $T_{2n-4} \leftarrow INT(G'_{[n]\setminus\{\alpha,\tau\}}, S)$;
  else if $\sigma(\tau) = 2$ **then**
    $M_\tau \leftarrow \{x', y'\}$, $M_\eta \leftarrow \{z'\}$;
    if $\sigma(\eta) = 1$ **then**
      $T_{2n-5} \leftarrow INT(G'_{\{\tau,\eta\}}, S)$,
      $T_{2n-4} \leftarrow INT(G'_{[n]\setminus\{\alpha,\tau,\eta\}}, S)$;
    else
      $\sigma(\eta) = 2$, $M_\eta \leftarrow \{z', x''\}$;
      if $\sigma(\eta') = 1$ and $\sigma(\theta') = 1$ **then**
        $M_{\eta'} \leftarrow \{z''\}$, $M_{\theta'} \leftarrow \{y''\}$,
        $T_{2n-5} \leftarrow INT(G'_{\{\tau,\eta'\}}, S)$,
        $T_{2n-4} \leftarrow INT(G'_{[n]\setminus\{\alpha,\tau,\eta'\}}, S)$;
      else
        $\sigma(\eta') = 2$, $M_{\eta'} \leftarrow \{z'', y''\}$,
        $u \leftarrow \{(z')', (z')''\} \setminus \{z\}$;
        if $u \notin V(BS^{\eta}_{n-1})$ **then**
          $T_{2n-5} \leftarrow INT(G'_{\{\eta,\eta'\}} \setminus \{z'\}, S)$,
          $T_{2n-4} \leftarrow INT(G'_{[n]\setminus\{\alpha,\eta,\eta'\}}, S)$;
        else
          $u \in V(BS^{\eta'}_{n-1})$, set $w \in N_{BS^\eta_{n-1}}(z')$ and $w' \notin V(BS^{\eta'}_{n-1})$,
          $T_{2n-5} \leftarrow INT(G'_{\{\eta,\eta'\}} \setminus \{z', w\}, S)$,
          $T_{2n-4} \leftarrow INT(G'_{[n]\setminus\{\alpha,\eta,\eta'\}}, S)$;
        end if
      end if
    end if
  else
    $\sigma(\tau) = 1$, $M_\tau \leftarrow \{x'\}$, $M_{\tau'} \leftarrow \{x''\}$, $M_\theta \leftarrow \{y'\}$, $M_{\theta'} \leftarrow \{y''\}$, $M_\eta \leftarrow \{z'\}$, $M_{\eta'} \leftarrow \{z''\}$, $T_{2n-5} \leftarrow INT(G'_{\{\tau,\theta,\eta\}}, S)$, $T_{2n-4} \leftarrow INT(G'_{\{\tau',\theta',\eta'\}}, S)$;
  end if
else if $\alpha = \beta \neq \gamma$ **then**
  Generate $2n-5$ internally disjoint (x, y)-paths $P_1, P_2, \cdots, P_{2n-5}$ in BS^α_{n-1} by Theorem 3.4 and Lemma 5.2;
  if neither of z' and z'' belong to BS^α_{n-1} **then**
    if $\ell(P_i) \ge 2$ for each $i \in [2n-5]$ **then**
      $Y \leftarrow \{x_i | x_i \in V(P_i) \setminus \{x, y\}$ and $1 \le i \le 2n-5\} \cup \{x, y\}$, $Y' \leftarrow \{u' | u \in Y\}$;
    else
      $\ell(P_i) = 1$ for some $i \in [2n-5]$, $P_1 \leftarrow P_i, x \leftarrow x_1, Y' \leftarrow \{u'|u \in Y\} \cup \{x''\}$; Generate $2n-3$ internally disjoint (z, Y')-paths $Q_1, Q_2, \cdots, Q_{2n-3}$ by Lemma 3.5 and Lemma 5.2;
      for $i = 1$ to $2n-5$ **do**
        $T_i \leftarrow P_i \cup Q_i \cup x_i x'_i$;
      end for
      $T_{2n-4} \leftarrow Q_{2n-4} \cup Q_{2n-3} \cup \{xx', yy'\}$ or $T_{2n-4} \leftarrow Q_{2n-4} \cup Q_{2n-3} \cup \{xx'', yy'\}$;
    end if
  else
    One of z' and z'' belong to BS^α_{n-1} and choose $z' \in V(BS^\alpha_{n-1})$;
    if $z' \notin V(P_i)$ **then**
      there is a (z', x)-path $\widetilde{P}$ in BS^α_{n-1}; set t be the first vertex in $\cup_{i \in [2n-5]} V(P)$ and $t \in V(P_{2n-5})$; $T_{2n-4} \leftarrow P_{2n-5} \cup \widetilde{P}[z', t] \cup zz'$;
    else
      $z' \in V(P_i)$, set $z' \in V(P_{2n-5}), T_{2n-4} \leftarrow P_{2n-5} \cup zz'$;
    end if
    if $\ell(P_i) \ge 2$ for each $i \in [2n-6]$ **then**
      $Y \leftarrow \{x_i | x_i \in V(P_i) \setminus \{x, y\}$ and $1 \le i \le 2n-6\} \cup \{x, y\}$, $Y' \leftarrow \{u' | u \in Y\}$;
    else
      $\ell(P_i) = 1$ for some $i \in [2n-6]$, $P_1 \leftarrow P_i, x \leftarrow x_1, Y' \leftarrow \{u'|u \in Y\} \cup \{x''\}$; Generate $2n-4$ internally disjoint (z, Y')-paths $Q_1, Q_2, \cdots, Q_{2n-4}$ by Lemma 3.5 and Lemma 5.2;
      for $i = 1$ to $2n-6$ **do**
        $T_i \leftarrow P_i \cup Q_i \cup x_i x'_i$;
      end for
      $T_{2n-5} \leftarrow Q_{2n-4} \cup Q_{2n-5} \cup \{xx', yy'\}$ or $T_{2n-5} \leftarrow Q_{2n-4} \cup Q_{2n-3} \cup \{xx'', yy'\}$;
    end if
  end if
else
  $\alpha \neq \beta, \beta \neq \gamma$ and $\alpha \neq \gamma$
  if $M \subseteq V(G_{\{\alpha,\beta,\gamma\}})$ **then**
    Generate $2n-4$ internally disjoint (x, y)-paths $P_1, P_2, \cdots, P_{2n-4}$ in $G_{\{\alpha,\beta\}}$ by Theorem 3.4 and

Lemma 5.2;
69: **if** $z' \notin \cup_{i\in[2n-4]}V(P_i)$ **then**
70: there is a (z', x)-path $\widetilde{P}$ in $G_{\{\alpha,\beta\}}$; set t be the first vertex in $\cup_{i\in[2n-4]}V(P_i)$ and $t \in V(P_{2n-4})$; $T_{2n-4} \leftarrow P_{2n-4} \cup \widetilde{P}[z', t] \cup \{zz'\}$;
71: **else**
72: $z' \in \cup_{i\in[2n-4]}V(P_i)$, set $z' \in V(P_{2n-4})$,
73: $T_{2n-4} \leftarrow P_{2n-4} \cup zz'$;
74: **end if**
75: $X \leftarrow \{x_i | x_i \in V(P_i) \cap N_{G_{\{\alpha,\beta\}}}(x)$ and $1 \le i \le 2n-5\}$, $X' \leftarrow \{x'_i | x_i \in X$ and $1 \le i \le 2n-5\}$; Generate $2n-5$ internally disjoint (z, X')-paths $Q_1, Q_2, \cdots, Q_{2n-5}$ in $G_{[n]\setminus\{\alpha,\beta\}}$ by Lemma 3.5 and Lemma 5.2;
76: **for** $i = 1$ to $2n-5$ **do**
77: $T_i \leftarrow P_i \cup Q_i \cup x_i x'_i$;
78: **end for**
79: **else**
80: $M \nsubseteq V(G_{\{\alpha,\beta,\gamma\}})$; set $z' \notin V(G_{\{\alpha,\beta\}})$;
81: **for** $i = 1$ to $2n-5$ **do**
82: Choose $x_i \in V(BS^{\alpha}_{n-1})$ and
83: $x'_i \in V(BS^{\beta}_{n-1})$;
84: **end for**
85: $X \leftarrow \{x_1, x_2, \cdots, x_{2n-5}\}$,
86: $X' \leftarrow \{x'_1, x'_2, \cdots, x'_{2n-5}\}$,
87: Generate $2n-5$ internally disjoint (x, X)-paths $P_1, P_2, \cdots, P_{2n-5}$ and $2n-5$ internally disjoint (y, X')-paths $P'_1, P'_2, \cdots, P'_{2n-5}$ by Lemma 3.5 and Lemma 5.2;
88: **for** $i = 1$ to $2n-5$ **do**
89: $\widehat{P}_i \leftarrow P_i \cup P'_i \cup x_i x'_i$;
90: **end for**
91: Set $Y \leftarrow \{x''_1, x''_2, \cdots, x''_{2n-5}\} \cup \{x'', y''\}$ with $x'', y'' \in V(G_{[n]\setminus\{\alpha,\beta\}})$; Generate $2n-3$ internally disjoint (z, Y)-paths $Q_1, Q_2, \cdots, Q_{2n-3}$ in $G_{[n]\setminus\{\alpha,\beta\}}$ by Lemma 3.5 and Lemma 5.2;
92: **for** $i = 1$ to $2n-5$ **do**
93: $T_i \leftarrow \widehat{P}_i \cup Q_i$,
94: **end for**
95: $T_{2n-4} \leftarrow Q_{2n-4} \cup Q_{2n-3} \cup \{xx'', yy''\}$
96: **end if**
97: **end if**

The explanation for Algorithm 1

Recall that $BS_n = BS^1_{n-1} \bigoplus BS^2_{n-1} \bigoplus \ldots \bigoplus BS^n_{n-1}$, where BS^i_{n-1} denotes the graph whose n-th bit number of any vertex is i and $i \in [n]$. Let $S = \{x, y, z\}$, where x, y and z are any three distinct vertices of BS_n. In line 1 of algorithm 1, we use α, β and γ to denote the n-th bit number of x, y and z, respectively.

If $\alpha = \beta = \gamma$, the vertices x, y and z belong to the same copy, BS^{α}_{n-1}, of BS_{n-1}. From line 2 to line 35, we give the method how to find $2n-4$ internally disjoint S-trees in BS_n;

If $\alpha = \beta \neq \gamma$, the vertices x, y and z belong to two different copies of BS_{n-1}, that is, x and y belong to the same copy of BS_{n-1} and z belong to the other copy of BS_{n-1}. From line 36 to line 65, we give the method how to find $2n-4$ internally disjoint S-trees in BS_n under this condition;

If any two of α, β and γ are not equal, that is, the vertices x, y and z belong to three different copies of BS_{n-1}. From line 66 to line 95, the method of how to find $2n-4$ internally disjoint S-trees in BS_n is given if $\alpha \neq \beta$, $\beta \neq \gamma$ and $\alpha \neq \gamma$.

6 Limitations of the work

In this paper, we introduce a network G_n that can be constructed recursively and contains exactly two outside neighbors. The network G_n contains many famous interconnection networks such as the alternating group graph AG_n, the k-ary n-cube Q^k_n, the split-star network S^2_n and the bubble-sort-star graph BS_n etc.. We mainly studied the generalized k-connectivity of the network G_n for $k = 3$, however, the generalized k-connectivity of G_n for $k \geq 4$ has not been studied. It would be an interesting and challenging work to study in the future.

7 Concluding remarks

The generalized k-connectivity is a generalization of the traditional connectivity. In this paper, we studied the generalized 3-connectivity of G_n that can be constructed recursively and contains exactly two outside neighbors. As applications of the main result, the generalized 3-connectivity of many famous networks such as the alternating group graph AG_n, the k-ary n-cube Q^k_n, the split-star network S^2_n and the bubble-sort-star graph BS_n can be obtained directly. In the future, we would like to study the generalized k-connectivity of G_n for $k \geq 4$, which would be interesting and challenging.

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