A Note on "A Tight Lower Bound on the Number of Channels Required for Deadlock-Free Wormhole Routing"

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Abstract—In [1], Libeskind-Hadas provided a tight lower bound on the number of channels required by a broad class of deadlock-free wormhole routing algorithms. In this short note, we show a simpler proof of the tight lower bound.

Index Terms—Deadlock-free routing, interconnection networks, strongly connected digraphs.

1 Introduction

In [1], Libeskind-Hadas provided a tight lower bound on the number of channels required by a deterministic or coherent deadlock-free routing algorithm. Given an interconnection network represented by a directed graph or a digraph $G = (N, C)$, where each vertex in N represents a node and each directed edge in C represents a unidirectional channel, Libeskind-Hadas showed that $|C| \geq 2|N| - 2$ is the tight lower bound on the number of channels required for deadlock-free wormhole routing. Note that a digraph may contain self-loops and multiple edges from one vertex to another, although a digraph for an interconnection network normally does not contain self-loops.

In this short note, we first review some basic concepts, present the problem, and finally provide a simpler proof of a major result in [1]. Given a digraph, two vertices, u and v, are said to be *strongly connected* if there exist directed paths from u to v and from v to u. $e = (u, v) \in C$ represents a directed edge from u to v. The existence of a deterministic or coherent adaptive deadlock-free wormhole routing in a given interconnection network $G = (N, C)$ is based on the following two requirements:

1. *Strongly connected requirement*: $G = (N, C)$ is strongly connected.
2. *Strictly decreasing path requirement*: There exists a deadlock-free labeling function $f : C \rightarrow \{1, 2, \ldots, |C|\}$ such that, for every pair of vertices u and v in G, there exists a path $p = v_0, v_1, \ldots, v_k$ such that $v_0 = u$ and $v_k = v$, and $f(v_{i-1}, v_i) > f(v_{j-1}, v_j)$ for $1 \leq i < j \leq k$.

Throughout this paper, $n = |N|$ represents the number of vertices in $G = (N, C)$ and $|C(G)|$, or simply $|C|$, represents the number of edges in G. For any vertex $v \in N$, we use $out(v) = \{(v, w) : w \in N\}$ to denote the set of outgoing edges at vertex v and use $in(v) = \{(u, v) : u \in N\}$ to denote the set of incoming edges at vertex v. Given a labeling function $f : C \rightarrow \{1, 2, \ldots, |C|\}$, for each vertex $v \in N$, let *max-out-label* of vertex v be $o(v) = \max\{f(e) : e \in out(v)\}$, and let *min-in-label* of vertex v be $i(v) = \min\{f(e) : e \in in(v)\}$. A directed path from u to v, with strictly decreasing labels (given by labeling function f) on the edges, is called a *strictly decreasing path* from u to v (with respect to f). Note that the labeling function does not need to be a one-to-one function. That is, it is possible that $f(e_1) = f(e_2)$ for $e_1 \neq e_2$.

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Lemma 1. *Let $G = (N, C)$ be a strongly connected digraph with n vertices, with $f : C \rightarrow \{1, 2, \cdots, |C|\}$ being a one-to-one labeling function on the edges. If $min\{o(v) : v \in N\} \geq n$, then $|C| \geq 2n - 1$.*

Proof. Notice that, for two different vertices u, v, $o(u) \neq o(v)$ because $out(u) \cap out(v) = \emptyset$. Hence, $\{o(v) : v \in N\}$ is a set of n distinct numbers, with each number being greater than or equal to n. Therefore, there exists a vertex v^* such that $o(v^*) = max\{o(v) : v \in N\} \geq 2n - 1$, which implies that $|C| \geq 2n - 1$. □

We now present a major result and then the result in [1].

Theorem 2. *Let $G = (N, C)$ be a strongly connected digraph with n vertices. Suppose there is a one-to-one labeling function $f : C \rightarrow \{1, 2, \cdots, |C|\}$ on the edges of G such that, for every pair of vertices u, v, there is a directed and strictly decreasing path from u to v. Then, $|C| \geq 2n - 2$.*

Proof. Use vertex t to denote the vertex with the *maximum min-in-label*. That is, $i(t) = \max\{i(v) : v \in C\}$. Then, $i(t) \geq n$ because $i(t)$ is the maximum over n distinct numbers. Since there is a strictly decreasing path from any other vertex to t in G, we can construct a spanning subgraph G_t of G rooted to t as follows: For every vertex $v \neq t$, find a directed and strictly decreasing path from v to t and put all the edges on the path into G_t.

Analogously, using s to denote the vertex with the *minimum max-out-label*, we can construct a spanning subgraph G_s of G rooted from s as follows: For every vertex $u \neq s$, find a directed decreasing path from s to u and put all the edges on the path into G_s.

Based on Lemma 1, we may assume that $o(s) \leq n - 1$ (otherwise, this theorem is proven). It is easy to see that $C(G_s) \cap C(G_t) = \emptyset$, otherwise, there would be an edge $(a, b) \in C(G_s) \cap C(G_t)$. This means that there is a directed and strictly decreasing path $p_1 = s, s_1, \cdots, a, b$ in G_s, and there is a directed and strictly decreasing path $p_2 = a, b, \cdots, t_1, t$ in G_t. Thus, we have

$$n - 1 \geq o(s) \geq f(s, s_1) \geq f(a, b) \geq f(t_1, t) \geq i(t) \geq n$$

which is a contradiction. Therefore,

$$|C| \geq |C(G_s)| + |C(G_t)| \geq (n - 1) + (n - 1) = 2n - 2.$$

□

Theorem 3 [1]. *Theorem 2 is still valid when the one-to-one labeling function f is replaced by a general labeling function g.*

Proof. Let $e_1, e_2, \cdots, e_{|C|}$ be an order of edges in a nondecreasing order of their edge labels based on g, i.e., $g(e_i) \leq g(e_j)$ for $i < j$. We define a one-to-one labeling function f such that $f(e_i) = i$. It is easy to see that, for any $i \neq j$, $g(e_i) < g(e_j)$ implies that $f(e_i) < f(e_j)$. Then, a strict decreasing path in G with respect to g is also a strict decreasing path in G with respect to f. Therefore, for every pair of vertices u, v in G, there is a directed and strictly decreasing path from u to v with respect to f. Based on Theorem 2, $|C| \geq 2n - 1$. □

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References

[1] R. Libeskind-Hadas, "A Tight Lower Bound on the Number of Channels Required for Deadlock-Free Wormhole Routing," *IEEE Trans. Computers*, vol. 47, no. 10, pp. 1,158-1,160, Oct. 1998.