On Two Allocation Methods For Web Services

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Abstract

In this paper, the properties of the multi-unit combinatorial auction (MUCA) and first-come-first-serve (FCFS) mechanisms for resource allocation in web services are disucssed. For the former mechanism, the web services provider (WSP) collects all the offers from the clients and then allocates the resource (via a greedy algorithm) to the clients in a way that maximizes the total return. For the later mechanism, the WSP simply allocates the resource to the client if the stock is available. Considering profit making, it is obvious that MUCA is a preferred mechanism. However, profit making is usually not the only consideration for mechanism selection. In practice, the number of clients who can get the resource and the volume of resource finally allocated are also important for management decision. By assuming that (a) the pricing curves of the clients are all identical and their marginal utility is decreasing, (b) resource being sold are divisible, (c) the number of units each client is willing to subscribe is uniformly distributed in $[0,1]$; and (d) the available resource is of constant size $(\bar{k})$; equations for the expected number of buyers who can get the product $(< b >)$ and the expected number of units being sold $(< s >)$ are derived analytically. Furthermore, by observing the numerical plots of these numbers against the number of clients, i.e. n, interesting results have been found. For MUCA, it is found that $< b >$ will be equal to n for $n \leq 2\bar{k}$ and approximately equal to $\frac{-1+\sqrt{1+8n\bar{k}}}{2}$ for $n \geq 2\bar{k}$. For the FCFS, it is found that $< b >$ will be equal to n for $n \leq 2\bar{k}$ and approximately equal to $\bar{k}$ for $n \geq 2\bar{k}$. Comparing the number of units being sold, i.e. $< s >$, it is found that the expected number of units being sold by either mechanism are approximately the same for $n \leq 2\bar{k}$. However, it is slightly larger in FCFS than MUCA if $n \geq 2\bar{k}$. These results can be used as a reference guideline for a web services provider to select appropriate mechanisms for allocating their computational resources to their clients.

1 Introduction

Extended from the ideas of software reuse and component based development, web services has been claimed to be a future direction for system development. Web services provider (WSP) makes application components available on the web. System developer can then integrate those components together by simply adding their URLs in the corresponding XML files. Once the application system is in production, it can invoke remotely the linked components and make use of the computational resources provided by the WSPs. Certainly, the usage of these resources is usually not free. Pricing mechanism becomes an important issue that every WSP needs to consider.

Mechanism selection is a notoriously difficult problem to any seller. Consider a WSP who has 20 servers[1] available for web services support. There are two problems that the WSP needs to consider : i) anticipate the number of buyers and their offers and ii) select among alternative mechanisms to allocate the servers. [2, 3] will be the obvious choice. Suppose the WSP[2] anticipates that about 8 clients will give offers :

Client	B1	B2	B3	B4	B5	B6	B7	B8
Servers	2	4	5	1	3	4	2	5
Price	10	30	35	6	15	18	12	35

and all of them can wait, multi-unit combinatorial auction (MUCA) will be the obvious choice. By using dynamic programming [4] to solve the following constraint optimization problem :

$$\begin{array}{ll} \text{Maximize} & 10s_1 + 30s_2 + 35s_3 + 6s_4 + 15s_5 \\ & +18s_6 + 12s_7 + 35s_8 \\ \text{Subject to} & 2s_1 + 4s_2 + 5s_3 + s_4 + 3s_5 + 4s_6 \\ & +2s_7 + 5s_8 \leq 20 \\ & s_i \in \{0,1\} \ \forall i = 1,\ldots,8, \end{array}$$

[1]Without loss of generality, the idea shown in this example can be applied to other computational resources, such as the CPU time and memory quota.

[2]In this paper, WSP and seller are used interchangably, so as buyer and client.

the WSP can anticipate that the profit is 133, by allocating all 20 servers to B2, B3, B4, B5, B7 and B8. Using greedy algorithm based on profit density [4], the profit will be 128, by allocating 19 servers to B1, B2, B3, B4, B7 and B8 with one server remains in the stock.

Now, the WSP faces another problem : *what should be the duration for the clients to submit their offers.* No client would like to wait forever. Therefore, it is rather risky if some clients withdraw their offers before the auction closed. Imagine that B2 withdraws the offer, the profit gain by using dynamic programming and greedy algorithm for allocation will reduce to 121 and 113 respectively.

To trade-off the risk, an alternative and yet simple mechanism is to allocate the servers to the clients in a first-come-first-serve (FCFS) basis. That is, once a client has given an offer, the WSP makes the deal if the number of servers is available. Suppose B1 is the first walk-in client, B2 is the second walk-in client and so on. B1, B2, B3, B4, B5 and B6 will finally be allocated and the WSP gets 114. In sequel, profit making should not be the only factor for mechanism selection. Some other factors, such as the number of clients who can eventually get the resources and the number of resources being allocated should also be considered.

In this paper, we will study analytically the expectation of these numbers in MUCA and FCFS. For MUCA, we focus on the profit density based greedy algorithm for resources allocation. We assume that a seller would like to sell $\bar{k}$ units of products. The seller estimates that there will be n customers interested in the product. The essential technique used in this paper is by using the formula derived by H. Weisberg [6] for a linear combination of order statistics and another formula derived by W. Feller (p.27 of [1]) for sum of uniformly random variables.

The rest of the paper will be organized as follows : The next section will describe the basic assumptions on the bid price and the bid size. The MUCA mechanism and the FCFS mechanism will be presented in Section 3. The expected number of customers and the expected number of product being sold for the mechanisms will be derived in Section 4. A discussion on the analytical results will also be presented in this section. The conclusion will be presented in Section 5. Due to page limit, all the proofs will be omitted here. Reader can refer to [5] for detail.

2 Assumptions

First, we assume that the bidders' valuation is a monotonically increasing function and its marginal utility is decreasing [2]. This is a reasonable assumption, since a buyer normally would like to have larger discount for larger purchase.

Assumption 1 *The pricing function of the buyers satisfy the following conditions : (i) (decreasing marginal utility)* $p'(k_i) \geq p'(k_j) \quad \forall 0 \leq k_i \leq k_j \leq 1$, *(ii)* $p(0) \geq 0$ *and* $p'(0) > 1$.

The following are some examples that satisfy the assumption for all $k \geq 0$: $f_1(k) = \alpha k + \beta$; $f_2(k) = \alpha \log(1+k) + \beta$, where $k \in [0,1]$, α and β are non-negative constant values. It should be noted that the bid price is a deterministic function dependent solely on the bid size. For the sake of analysis, it is necessary to show that the following property for a decreasing marginal valuation function holds.

Theorem 1 *For any non-negative real-valued function* $f(x)$ *satisfies that* $f'(x) \geq f'(y) \geq 0$ *for all* $0 \leq x \leq y$ *and* $f(0) \geq 0$, *then the following conditions hold : (i)* $\frac{f(x)}{x} \geq f'(x)$; *(ii)* $\frac{f(x)}{x} \geq \frac{f(y)}{y}$, *for all* $0 \leq x \leq y$.

Second, we assume that the items are divisible; i.e. bidder can bid for any fractional number of items. Without loss of generality, we assume that the maximum bid size is one.

Assumption 2 *The items are divisible.*

Assumption 3 *The bid sizes are random variables from a uniform distribution, i.e.* k_i *is a random variable from* $U(0,1)$.

3 Methods for allocation

In either selling mechanism, it should be noted that the bid size is a random variable from a uniform distribution between zero and one, while the bid price is a deterministic function dependent on the bid size. The number of customers/bidders n is information that the seller estimates in advance.

3.1 Greedy Method

Suppose there are n bidders whose bid prices and bid sizes are $p_1, \ldots, p_n$ and $k_1, \ldots, k_n$ respectively. Once all the sealed bids have been collected, the seller can apply the algorithm shown in Figure 3.1 to determine the allocation. First, the bids are ranked in descending order of their profit density, i.e.

$$\frac{p_{1:n}}{k_{1:n}} \geq \frac{p_{2:n}}{k_{2:n}} \geq \ldots \geq \frac{p_{n:n}}{k_{n:n}}. \tag{1}$$

Then, we allocate the units to the first b bidders, such that

$$\sum_{i=1}^{b} k_{i:n} \quad \leq \quad \bar{k} \tag{2}$$

$$\sum_{i=1}^{b+1} k_{i:n} \quad > \quad \bar{k}. \tag{3}$$

```
WAITFOR (w_i, p_i), i = 1, ..., n;
SORT {p_i / w_i} s.t. p_{i:n}/w_{i:n} <= p_{j:n}/w_{j:n}  ∀ i <= j;
SET C = k̄;
SET P = 0;
SET j = 1;
WHILE(C - w_{j:n} > 0 and j <= n)
   C = C - w_{j:n};
   P = P + p_{j:n};
   j = j + 1;
END
```

Figure 1: Greedy method for allocation.

```
SET C = k̄;
SET P = 0;
SET j = 1;
WHILE(C - w_j > 0 and j <= n)
   C = C - w_j;
   P = P + p_j;
   j = j + 1;
END
```

Figure 2: FCFS method for allocation.

In accordance with the assumption that the valuation functions of all the bidders are the same and their marginal valuations are decreasing, Equation (1) implies that $k_{1:n} \leq k_{2:n} \leq \ldots \leq k_{n:n}$. We can therefore rank the bidders according to their bid sizes. Again, the $\bar{k}$ units are allocated to the first b bidders according to conditions (2) and (3).

3.2 First-come-first-serve

FCFS method is similar to conventional retail. Once a customer has walked into the shop, the seller simply sells the product to the customer until the inventor is zero. The algorithm is shown in Figure 3.2. Compared with the greedy method, FCFS does not need to wait until all the n customers have arrived and then sort their profit densities. In terms of computational complexity, the greedy method requires $\mathcal{O}(n \log n)$ steps for sorting and $\mathcal{O}(n)$ memory size for storing the (w_i, p_i) pairs. On the other hand, the FCFS method requires only $\mathcal{O}(n)$ computation steps and $\mathcal{O}(1)$ memory size.

4 Analysis

For the sake of analysis, let $S(w,n) = \sum_{i=1}^{w} k_{i:n}$ be the sum of units being sold to the first w bidders on the ranking list of n bidders. Similarly, we let $S_r(w,n) = \sum_{i=1}^{w} k_i$ be the sum of units being sold to the first w (first-come-first-serve) customers from a queue of n walk-in customers.

4.1 Greedy allocation method

Using the Weisberg formula [6], evaluate $Pr\{S(w,n) \leq \bar{k}\}$ by setting

$$d_i = \begin{cases} 1 & \forall i = 1, \ldots, w \\ 0 & \forall i = w+1, \ldots, n. \end{cases}$$

For the case that exactly w winners are allocated product in an auction, i.e.

$$\{S(w,n) \leq \bar{k} \text{ and } S(w+1,n) > \bar{k}\},$$

we consider the following events:

$$\begin{aligned} E_1 &= \{S(w,n) \leq \bar{k} \text{ and } S(w+1,n) \leq \bar{k}\} \\ E_2 &= \{S(w,n) \leq \bar{k} \text{ and } S(w+1,n) > \bar{k}\} \\ E_3 &= \{S(w,n) > \bar{k} \text{ and } S(w+1,n) \leq \bar{k}\} \\ E_4 &= \{S(w,n) > \bar{k} \text{ and } S(w+1,n) > \bar{k}\}. \end{aligned}$$

Probabilities for the events E_1, E_3, E_4 can readily be determined as follows :

$$\begin{aligned} Pr\{E_1\} &= Pr\{S(w+1,n) \leq \bar{k}\} \\ Pr\{E_3\} &= 0 \\ Pr\{E_4\} &= 1 - Pr\{S(w,n) \leq \bar{k}\}. \end{aligned}$$

4.1.1 Expected No. of winners $\langle b \rangle$

Using the fact that $Pr\{E_1\} + Pr\{E_2\} + Pr\{E_3\} + Pr\{E_4\} = 1$, the probability of exactly w winners in an auction can be determined as follows :

$$Pr\{w \text{ winners}\} = \begin{cases} \Delta Pr(w) & \text{if } w < n \\ Pr\{S(n,n) \leq \bar{k}\} & \text{if } w = n. \end{cases}, \tag{4}$$

$$\Delta Pr(w) = Pr\{S(w,n) \leq \bar{k}\} - Pr\{S(w+1,n) \leq \bar{k}\}.$$

This equation applies for all $w \geq \bar{k}$ and the evaluation of the $Pr\{S(w.n) \leq \bar{k}\}$ can be based on the Weisberg formula. Thus, the expected number of winners in an auction, $\langle b \rangle$ can be determined by the following formula.

$$\begin{aligned} \langle b \rangle &= \sum_{w=1}^{n-1} w \left(Pr\{S(w,n) \leq \bar{k}\} - Pr\{S(w+1,n) \leq \bar{k}\} \right) \\ &+ nPr\{S(n,n) \leq \bar{k}\}. \end{aligned} \tag{5}$$

Since $Pr\{b = w\} = 1$ for all $w \leq \bar{k}$, the above equation can be rewritten as follows :

$$\begin{aligned} \langle b \rangle &= \sum_{w=\bar{k}}^{n-1} w \left(Pr\{S(w,n) \leq \bar{k}\} - Pr\{S(w+1,n) \leq \bar{k}\} \right) \\ &+ nPr\{S(n,n) \leq \bar{k}\}. \end{aligned} \tag{6}$$

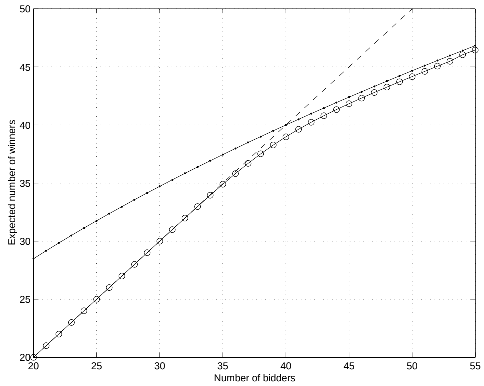

Figure 3: The expected number of winners against the number of bidders in an auction. Here the number of units for sale is 20.

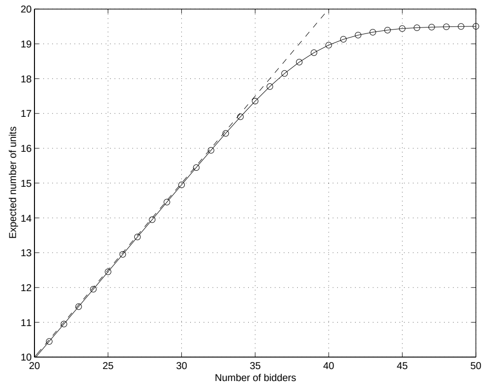

Figure 4: The expected number of units being sold against the number of bidders in an auction. Here the number of units for sale is 20.

It is a function dependent on n and $\bar{k}$. Once n and $\bar{k}$ are known, $\langle b \rangle$ can be evaluated numerically.

Figure 3 illustrates the case when $\bar{k} = 20$. We have also plotted the curve for the cases when $\bar{k}$ equals to 30 and 40 respectively. All show the same behavior. For $n \leq 2\bar{k}$ and $n \geq 2\bar{k}$, $\langle b \rangle$ can be approximated as follows :

$$\langle b \rangle \approx \begin{cases} n & \text{if } n \leq 2\bar{k} \\ \frac{-1+\sqrt{1+8n\bar{k}}}{2} & \text{if } n \geq 2\bar{k}. \end{cases} \tag{7}$$

4.1.2 Expected No. of units being sold $\langle S \rangle$

The expected number of units being sold can thus be evaluated by using similar argument. First, let us consider the event exactly w winners in the auction and x number of units being sold, i.e. $\{S(w,n) \leq x$ and $S(w+1,n) \geq \bar{k}\}$. Obviously, $\bar{k} - 1 \leq x \leq \bar{k}$. Since $\{S(w,n) \leq x\}$ equals

$$\{S(w,n) \leq x \text{ and } S(w+1,n) \leq \bar{k}\} \cup \{S(w,n) \leq x \text{ and } S(w+1,n) \geq \bar{k}\},$$

and the first event is equivalent to $\{S(w+1,n) \leq \bar{k}\}$, it is readily shown that

$$\begin{aligned} & Pr\{S(w,n) \leq x \text{ and Exactly } w \text{ winners}\} \\ = \; & Pr\{S(w,n) \leq x \text{ and } S(w+1,n) \geq \bar{k}\} \\ = \; & Pr\{S(w,n) \leq x\} - Pr\{S(w+1,n) \leq \bar{k}\} \end{aligned} \tag{8}$$

for all $x \in \{y | Pr\{S(w,n) \leq y\} - Pr\{S(w+1,n) \leq \bar{k}\} \geq 0\}$. Let $h(x|w,n,\bar{k})$ be the probability $\{S(w,n) = x\}$ given that {Exactly w winners}. It can thus be evaluated as follows :

$$h(x|w,n,\bar{k}) = \begin{cases} \frac{d}{dx}\frac{\Delta Pr(w,x)}{\Delta Pr(w)} & \text{if } w < n \\ \frac{d}{dx}\frac{Pr\{S(n,n) \leq x\}}{Pr\{S(n,n) \leq \bar{k}\}} & \text{if } w = n. \end{cases} \tag{9}$$

for all $x \in \{y | Pr\{S(w,n) \leq y\} - Pr\{S(w+1,n) \leq \bar{k}\} \geq 0\}$.

$$\Delta Pr(w,x) = Pr\{S(w,n) \leq x\} - Pr\{S(w+1,n) \leq \bar{k}\}.$$

The expected number of units being sold, $\langle S \rangle$, can thus be written as follows :

$$\begin{aligned} \langle S \rangle \; = \; & \sum_{w=\bar{k}}^{n-1} \int_{x_w}^{\bar{k}} x d\left(\Delta Pr(w,x)\right) \\ & + \int_0^{\bar{k}} x d Pr\{S(n,n) \leq x\}, \end{aligned} \tag{10}$$

for all $n \geq \bar{k}$ and x_w satisfies the condition :

$$Pr\{S(w,n) \leq x_w\} = Pr\{S(w+1,n) \leq \bar{k}\}.$$

Figure 4 shows the case when $\bar{k}$ equals to 20.

For large n, an approximated equation for the expected number of units being sold can be derived. Considering the residue, $R(n, \langle b \rangle, \bar{k}) = \bar{k} - \sum_{i=1}^{\langle b \rangle} \frac{i}{n}$ satisfies the following inequality : $0 \leq R(n, \langle b \rangle, \bar{k}) \leq (\langle b \rangle + 1/n)$ and supposing that this residue is uniform distributed on $[0, (\langle b \rangle + 1)/n]$. The expected residue, $\langle R \rangle$ can be written as follows : $\langle R \rangle = (\langle b \rangle + 1/2n) = (\bar{k}/\langle b \rangle)$. Substituting the approximation for $\langle b \rangle$ in Equation (7), the approximation of the expected number of units being sold can be written as follows :

$$\langle S \rangle \approx \bar{k} \left(\frac{\sqrt{1+8n\bar{k}} - 3}{\sqrt{1+8n\bar{k}} - 1} \right) \tag{11}$$

for $n \gg \bar{k}$.

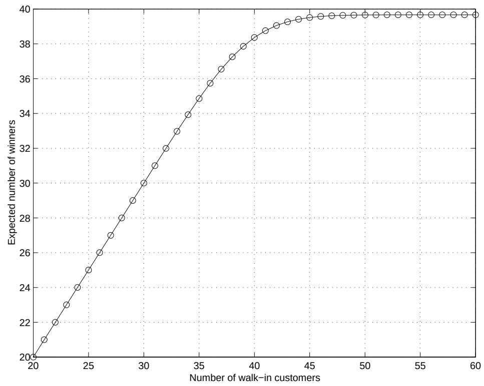

Figure 5: The expected number of customers who can get the products against the number of walk-in customers. Here the number of units for sale is 20.

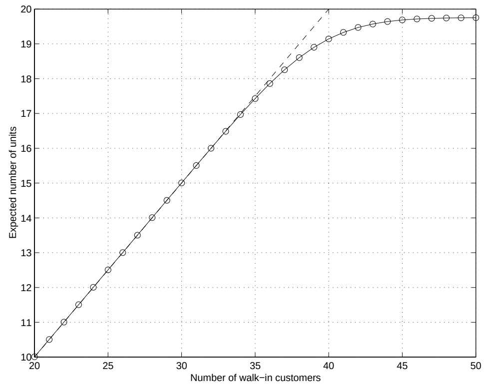

Figure 6: The expected number of units being sold against the number of walk-in customers. Here the number of units for sale is 20.

4.2 First-come-first-serve

For the case that the products are allocated in a first-come-first-serve basis, we consider the following equation : $S_r(w,n) = \sum_{i=1}^{w} k_i$, for all $\bar{k} \leq w \leq n$. By replacing $S(w,n)$ by $S_r(w,n)$, we can use the same argument used for greedy method to derive the equations for the expected number of customers, $\langle b_r \rangle$ and the expected number of units being sold $\langle S_r \rangle$.

4.2.1 Expected No. of customers $\langle b_r \rangle$

The expected number of customers, $\langle b_r \rangle$, can be determined by the following formula.

$$\langle b_r \rangle = \sum_{w=\bar{k}}^{n-1} w \Delta Pr(w) + nPr\{S_r(n,n) \leq \bar{k}\}. \quad (12)$$

The experession for $Pr\{S_r(w,n) \leq x\}$ will be from Feller formula [1]. Figure 5 shows the case that the number of units for sale is 20.

4.3 Expected number of units being sold $\langle S_r \rangle$

The expected number of units being sold $\langle S_r \rangle$ can be determined by the following equation.

$$\begin{aligned} \langle S_r \rangle &= \sum_{w=\bar{k}}^{n-1} \int_{x_w}^{\bar{k}} xd\left(\Delta Pr(w,x)\right) \\ &+ \int_{0}^{\bar{k}} xdPr\{S_r(n,n) \leq x\}, \end{aligned} \quad (13)$$

for all $n \geq \bar{k}$ and x_w satisfies the condition : $Pr\{S_r(w,n) \leq x_w\} = Pr\{S_r(w+1,n) \leq \bar{k}\}$. Figure 6 shows the case when the number of units on sale is 20. It should be noted that the number of units being sold by the first-come-first-serve basis is slightly higher than that being sold by auction, Figure 7.

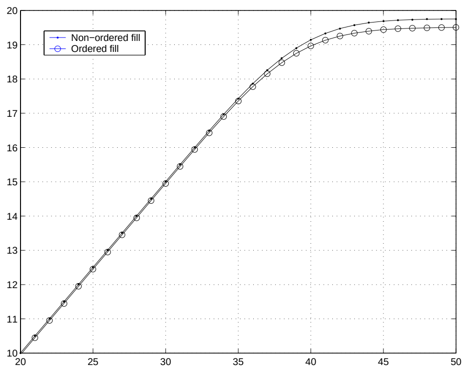

Figure 7: Comparison between FCFS and the greedy method in terms of the expected number of units being sold. Here the number of units for sale is 20.

4.4 Discussion

The results obtained in this section are summarized in Table 1.

- It is found that the only difference between auction and FCFS selling is when $n \gg 2\bar{k}$. The auction method provides products to more customers. This can eventually let the seller increase his customer base.
- Consider the payment function as follows : $p(k) = \alpha k + \beta$, where $\alpha > 0$ and $\beta > 0$. Here, β can be treated as the premium (e.g. membership fee)

	Auction	FCFS
$n \le 2\bar{k}$	$\langle b \rangle \approx n$ $\langle S \rangle \approx \frac{n}{2}$	$\langle b_r \rangle \approx n$ $\langle S_r \rangle \approx \frac{n}{2}$
$n \gg 2\bar{k}$	$\langle b \rangle = \frac{-1+\sqrt{1+8n\bar{k}}}{2}$ $\langle S \rangle \approx \bar{k}$	$\langle b_r \rangle \approx \bar{k}$ $\langle S_r \rangle \approx \bar{k}$

Table 1: Summary on the expected number of customers and the expected number of units being sold for both auction and FCFS.

that the customer may or may not have to pay in advance.

- Also, it can be assumed that only the winners have to pay the premium (e.g. administration cost). The expected revenues that the seller can gain from auction, $\langle P \rangle$, and FCFS, $\langle P_r \rangle$, can be written as follows :

$$\text{MUCA:} \quad \langle P \rangle = \alpha \langle S \rangle + \beta \langle b \rangle; \quad (14)$$
$$\text{FCFS:} \quad \langle P_r \rangle = \alpha \langle S_r \rangle + \beta \langle b_r \rangle. \quad (15)$$

  Since $\langle b \rangle$ is larger than $\langle b_r \rangle$ and $\langle S \rangle$ is slightly smaller than $\langle S_r \rangle$, the expected revenue that the seller gains from auction should be larger than FCFS.

- In the case when all the bidders have to pay the premium, the revenue will be $\alpha \langle b \rangle + \beta n$. Its value can be quite large for large n.

• If $\beta = 0$, the expected revenue that the seller gains from FCFS selling will be larger than from auction. If $\alpha = 0$, the expected revenue that the seller gains from auction will be larger than from FCFS selling.

• Without loss of generality, the results obtain in this paper can be extended for the case when k_i is uniformly distributed on the range $[0, M]$. Considering the following problem :

$$\begin{array}{ll} \text{Maximize} & \sum_{i=1}^{n} p_i s_i \\ \text{Subject to} & \sum_{i=1}^{n} k_i s_i \le N \\ & s_i \in \{0,1\} \ \forall i = 1, \ldots, n. \end{array}$$

Here N is the total number of units being auctioned off. Dividing both sides of the inequality constraint by M, the following inequality obtained, $\sum_{i=1}^{n} (k_i/M) s_i \le (N/M)$. Noting that $k_i/M \in [0,1]$ and $N/M \in [0,1]$. Hence, the results summarized in Table 1 can easily be extended for this general problem, as depicted in Table 2.

	Auction	FCFS
$n \le 2N/M$	$\langle b \rangle \approx n$ $\langle S \rangle \approx \frac{n}{2}$	$\langle b_r \rangle \approx n$ $\langle S_r \rangle \approx \frac{n}{2}$
$n \gg 2N/M$	$\langle b \rangle = \frac{-1+\sqrt{1+8nN/M}}{2}$ $\langle S \rangle \approx N$	$\langle b_r \rangle \approx N/M$ $\langle S_r \rangle \approx N$

Table 2: Summary on the expected number of customers and the expected number of units being sold for both auction and FCFS. N and M (> 1) correspond to the total number of units for sale and the $\max\{k_i\}$.

5 Conclusion

Selection of selling mechanism among various alternatives is a difficult problem in business. The major contribution of this paper is making use of two formulae from statistics to analyze numerically the statistical behavior of multi-units combinatorial auction and comparing the results with first-come-first-serve mechanism. Observed from the numerical plots, it is found that the expected number of customers and the expected number of units being sold are almost no different between auction and FCFS sell. This result can help the WSP to select appropriate mechanism for leasing web services. If the expected number of clients who are interested in the services is less than $2nN/M$, where N and M are the number of quota in stock and the maximum number of units that a client would like to subscribe, the WSP should simply rent the service via FCFS sell; otherwise, the WSP should try auction.

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